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On the Hasse-Sondow Formula for the Riemann Zeta Function

C. G. León-Vega and J. López-Bonilla

ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5, 1er. Piso,
Col. Lindavista CP 07738, CDMX, México

Abstract: We employ the known relation $(2^{1-s}-1)\zeta(s)=\sum_{r=1}^{\infty}\frac{(-1)^r}{r^s}$ to motivate the analytic continuation of the Riemann zeta function obtained by Hasse and Sondow.

Key words: Hasse-Sondow's expression · Analytic continuation of $\zeta(s)$.

INTRODUCTION

We have the following relation for the Riemann zeta function [1]:

$$\zeta(s)=\frac{1}{2^{1-S}-1}\sum_{r=1}^{\infty}\frac{(-1)^r}{r^s},\quad 0>\text{Re } s<1, \tag{1}$$

for example [2, 3]:

$$\zeta\left(\frac{1}{2}=\right)(1+\sqrt{2})\sum_{r=1}^{\infty}\frac{(-1)^r}{\sqrt{r}}=-1.4603\ 5450\ 88 \tag{2}$$

In Sec. 2 we manipulate (1) to obtain the analytic continuation of $\zeta(s)$ deduced by Hasse [4] and Sondow [5, 6], which allows determine it $\forall\, s\neq 1$

Hasse-Sondow's formula

From (1):

$$(1-2^{1-s})\zeta(s)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^s}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^s}\frac{1}{2^{k+1}}1F_0\left(k+1;\frac{1}{2}\right), \tag{3}$$

with the presence of the hypergeometric function [7]:

$$1F_0(a;z)=\frac{1}{(1-z)^a},\quad |Z|<1\quad \therefore\quad 1F_0\left(k+1;\frac{1}{2}\right)=2^{k+1}. \tag{4}$$

It is simple to see that:

$$\sum_{n=k}^{\infty}\binom{n}{k}\frac{1}{2^{n-k}}=\sum_{q=0}^{\infty}\binom{q+k}{k}\frac{1}{2^q}=1F_0\left(k+1;\frac{1}{2}\right), \tag{5}$$

Then (3) implies the relation:

$$(1-2^{1-s})\zeta(s)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^s}\sum_{n=k}^{\infty}\binom{n}{k}\frac{1}{2^{n+1}},$$

That is [8, 9]:

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{-s} \tag{6}$$

obtained by Hasse [4] and Sondow [5, 6] for arbitrary $s\neq 1$.

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Corresponding Author: C.G. León-Vega, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5, 1er. Piso, Col. Lindavista CP 07738, CDMX, México.

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