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A Brief Study on Some Expansion Formulae for the $\overline{H}$-function

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Abstract: In the present paper, the author has established two expansion formula of $\overline{H}$-Function.

Key words: $\overline{H}$-Function · Expansion Formula · Gamma Function
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INTRODUCTION

The $\overline{H}$-function occurring in the paper will be defined and represented as follows:

$$\overline{H}_{P,Q}^{M,N}[z] = \overline{H}_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right.\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \overline{\phi}(\xi) z^{\xi} d\xi \tag{1.1}$$

where

$$\overline{\phi}(\xi) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j\xi) \prod_{j=1}^{N} \left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q} \left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j} \prod_{j=N+1}^{P} \Gamma(a_j-\alpha_j\xi)} \tag{1.2}$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j (j = 1,....,p)$ and $b_j (j = 1,....,Q)$ are complex parameters, $\alpha_j \ge 0 (j = 1,...,P), \beta_j \ge 0 (j = 1,...,Q)$ (not all zero simultaneously) and exponents $A_j (j = 1,....,N)$ and $B_j (j = N + 1,....,Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$-function given by equation (1.1) have been given by (Buschman and Srivastava[1]).

$$\Omega \equiv \sum_{j=1}^{M} |\beta_j| + \sum_{j=1}^{N} |A_j\alpha_j| - \sum_{j=M+1}^{Q} |\beta_j B_j| - \sum_{j=N+1}^{P} |\alpha_j| > 0 \tag{1.3}$$

and

$$|\arg(z)| < \frac{1}{2}\pi\Omega \tag{1.4}$$

If we take $A_j = 1 (j = 1,...,N), B_j = 1 (j = M+1,...,Q)$ in (1.1), the function $\overline{H}_{P,Q}^{M,N}$ reduces to the Fox's H-function [2]. We shall use the following notation:

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$$A^{*}=\left(a_{j},\alpha_{j};A_{j}\right)_{1,N},\left(a_{j},\alpha_{j}\right)_{N+1,P}\ B^{*}=\left(b_{j},\beta_{j}\right)_{1,M},\left(b_{j},\beta_{j};B_{j}\right)_{M+1,Q},$$

Expansion Formula

First Formula:

$$\begin{aligned}(a_p-\mu a_1)\overline{H}_{p,q}^{m,n}&\left[z\left|\begin{matrix}(a_1+1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(1+a_p,\mu\sigma)\\ B^*\end{matrix}\right.\right]\\ &=\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1+1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(a_p,\mu\sigma)\\ B^*\end{matrix}\right.\right]\\ &+\mu\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(1+a_p,\mu\sigma)\\ B^*\end{matrix}\right.\right]\end{aligned}\tag{2.1}$$

where $1\le n\le p-1,\mu>0$.

Proof: We have to prove that

$$\begin{aligned}(a_p-\mu a_1)\overline{H}_{p,q}^{m,n}&\left[z\left|\begin{matrix}(a_1+1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(1+a_p,\mu\sigma)\\ B^*\end{matrix}\right.\right]\\ &-\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1+1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(a_p,\mu\sigma)\\ B^*\end{matrix}\right.\right]\\ &-\mu\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(1+a_p,\mu\sigma)\\ B^*\end{matrix}\right.\right]=0\end{aligned}$$

$\therefore$ L.H.S.=

$$\begin{aligned}&(a_p-\mu a_1)\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}\Gamma(-a_1+\sigma s)}{\prod_{j=n+1}^{p-1}\Gamma(a_j-\alpha_j s)\prod_{j=m+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\Gamma(a_p+1-\mu\sigma s)}z^s ds\\ &-\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}\Gamma(-a_1+\sigma s)}{\prod_{j=n+1}^{p-1}\Gamma(a_j-\alpha_j s)\prod_{j=m+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\Gamma(a_p-\mu\sigma s)}z^s ds\\ &-\mu\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}\Gamma(1-a_1+\sigma s)}{\prod_{j=n+1}^{p-1}\Gamma(a_j-\alpha_j s)\prod_{j=m+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\Gamma(a_p+1-\mu\sigma s)}z^s ds\end{aligned}$$

Let $\bar{\phi}(s)=\dfrac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}}{\prod_{j=n+1}^{p-1}\Gamma(a_j-\alpha_j s)\prod_{j=m+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}}$

L.H.S.= $\frac{1}{2\pi i}\int_L \bar{\phi}(s)\frac{(ap-\mu a_1)\Gamma(-a_1+\sigma s)}{(a_p-\mu\sigma s)\Gamma(a_p-\mu\sigma s)}z^s ds$

$$-\frac{1}{2\pi i}\int_L \bar{\phi}(s)\frac{\Gamma(-a_1+\sigma s)}{\Gamma(a_p-\mu\sigma s)}z^s ds-\mu\frac{1}{2\pi i}\int_L \bar{\phi}(s)\frac{(-a_1+\sigma s)\Gamma(-a_1+\sigma s)}{(a_p-\mu\sigma s)\Gamma(a_p-\mu\sigma s)}z^s ds$$

$$=\frac{1}{2\pi i}\int_L \bar{\phi}(s)\frac{\Gamma(-a_1+\sigma s)}{(a_p-\mu\sigma s)\Gamma(a_p-\mu\sigma s)}\left[a_p-\mu a_1-a_p+\mu\sigma s+\mu a_1-\mu\sigma s\right]z^s ds$$

$$=\frac{1}{2\pi i}\int_L \bar{\phi}(s)\frac{\Gamma(-a_1+\sigma s)}{(a_p-\mu\sigma s)\Gamma(a_p-\mu\sigma s)}[0]z^s ds=0=\text{R.H.S.}$$

Second Formula:

$$(a_p-\mu a_1)\overline{H}^{m+1,n}_{p+1,q+1}\left[z\left|\begin{matrix}(a_1+1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(1+a_p,\mu\sigma),(a_1+\nu,\sigma)\\(1+a_1+\nu,\sigma),B^*\end{matrix}\right.\right]$$
$$=\nu\overline{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1+1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(a_p,\mu\sigma)\\B^*\end{matrix}\right.\right]$$
$$+(\mu\nu-a_p+\mu a_1)\overline{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(1+a_p,\mu\sigma)\\B^*\end{matrix}\right.\right] \tag{2.2}$$

where $1\le n\le p-1,\mu>0$.

Proof: We have to prove that

$$(a_p-\mu a_1)\overline{H}^{m+1,n}_{p+1,q+1}\left[z\left|\begin{matrix}(a_1+1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(1+a_p,\mu\sigma),(a_1+\nu,\sigma)\\(1+a_1+\nu,\sigma),B^*\end{matrix}\right.\right]$$
$$-\nu\overline{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1+1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(a_p,\mu\sigma)\\B^*\end{matrix}\right.\right]$$
$$-(\mu\nu-a_p+\mu a_1)\overline{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,\sigma;1),(a_j,\alpha_j;A_j)_{2,n},(a_j,\alpha_j)_{n+1,p-1},(1+a_p,\mu\sigma)\\B^*\end{matrix}\right.\right]=0$$

$\therefore$ L.H.S.=

$$(a_p-\mu a_1)\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}\Gamma(-a_1+\sigma s)\Gamma(1+a_1+\nu-\sigma s)}{\prod_{j=n+1}^{p-1}\Gamma(a_j-\alpha_j s)\prod_{j=m+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\Gamma(a_p+1-\mu\sigma s)\Gamma(a_1+\nu-\sigma s)}z^s ds$$
$$-\nu\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}\Gamma(1-1-a_1+\nu-\sigma s)}{\prod_{j=n+1}^{p-1}\Gamma(a_j-\alpha_j s)\prod_{j=m+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\Gamma(a_p-\mu\sigma s)}z^s ds$$
$$-(\mu\nu-a_p+\mu a_1)\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=2}^{n}\left\{\Gamma(1-a_j+\alpha_j s)\right\}^{A_j}\Gamma(1-a_1+\sigma s)}{\prod_{j=n+1}^{p-1}\Gamma(a_j-\alpha_j s)\prod_{j=m+1}^{q}\left\{\Gamma(1-b_j+\beta_j s)\right\}^{B_j}\Gamma(1+a_p-\mu\sigma s)}z^s ds$$

Let $\bar{\phi}(s) = \dfrac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=2}^{n} \left\{\Gamma(1 - a_j + \alpha_j s)\right\}^{A_j}}{\prod_{j=n+1}^{p-1} \Gamma(a_j - \alpha_j s) \prod_{j=m+1}^{q} \left\{\Gamma(1 - b_j + \beta_j s)\right\}^{B_j}}$

L.H.S.= $(a_p - \mu a_1)\dfrac{1}{2\pi i}\displaystyle\int_L \bar{\phi}(s)\dfrac{\Gamma(-a_1+\sigma s)\Gamma(1+a_1+\nu-\sigma s)}{(a_p-\mu\sigma s)\Gamma(a_p-\mu\sigma s)\Gamma(a_1+\nu-\sigma s)}z^s ds$

$$-\nu\frac{1}{2\pi i}\int_L \bar{\phi}(s)\frac{\Gamma(-a_1+\sigma s)}{\Gamma(a_p-\mu\sigma s)}z^s ds - (\mu\nu - a_p + \mu a_1)\frac{1}{2\pi i}\int_L \bar{\phi}(s)\frac{(-a_1+\sigma s)\Gamma(-a_1+\sigma s)}{(a_p-\mu\sigma s)\Gamma(a_p-\mu\sigma s)}z^s ds$$

$$= \frac{1}{2\pi i}\int_L \bar{\phi}(s)\frac{\Gamma(-a_1+\sigma s)}{\Gamma(a_p-\mu\sigma s)}\left[\frac{(a_1+\nu-\sigma s)(a_p-\mu a_1)}{(a_p-\mu\sigma s)} - \nu - \frac{(\mu\nu - a_p+\mu a_1)(-a_1+\sigma s)}{(a_p-\mu\sigma s)}\right]z^s ds$$

$$= \frac{1}{2\pi i}\int_L \bar{\phi}(s)\frac{\Gamma(-a_1+\sigma s)}{\Gamma(a_p-\mu\sigma s)(a_p-\mu\sigma s)}$$

$$\left[(a_1+\nu-\sigma s)(a_p-\mu a_1) - \nu(a_p-\mu\sigma s) - (\mu\nu - a_p + \mu a_1)(-a_1+\sigma s)\right]z^s ds$$

$$= \frac{1}{2\pi i}\int_L \bar{\phi}(s)\frac{\Gamma(-a_1+\sigma s)}{\Gamma(a_p-\mu\sigma s)(a_p-\mu\sigma s)}$$

$$\left[a_1 a_p - \sigma s a_p - a_1^2\mu - \mu\nu a_1 + \mu\sigma s a_1 + \mu\sigma s\nu + a_1\mu\nu - a_1 a_p + \sigma s a_p + a_1^2\mu - \mu\sigma s a_1 - \mu\sigma s\nu\right]ds$$

$$= \frac{1}{2\pi i}\int_L \bar{\phi}(s)\frac{\Gamma(-a_1+\sigma s)}{\Gamma(a_p-\mu\sigma s)(a_p-\mu\sigma s)}[0]ds = 0 \text{=R.H.S.}$$

For $A_j = 1(j=1,\ldots,n),\ B_j = 1(j=m+1,\ldots,q)$ in (2.1),(2.2), we get the results in terms of Fox's H-function [2].

REFERENCES

1. Buschman, R.G. and H.M. Srivastava, 1990. The $\overline{H}$ function associated with a certain class of Feynman integrals, J. Phys. A: Math. Gen., 23: 4707-4710.
2. Fox, C., 1961. The *G*-function and *H*function as symmetric Fourier kernels, Trans. Amer. Math. Soc. 98: 396-429.
3. Srivastava, H.M., K.C. Gupta and S.P. Goyal, 1982. The H-Functions of One and Two Varibale with Applications, South Asian Publishers, New Delhi and Madras.