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Two Special Values for Hypergeometric Functions

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Abstract: We deduce the hypergeometric version of two combinatorial identities obtained by Nemes *et al*. Besides, we give a simple proof for an identity involving harmonic numbers.

Key words: Hypergeometric functions · Harmonic numbers · Binomial coefficients

INTRODUCTION

In [1] we find the following identities:

$$A \equiv \sum_{k=1}^{n} \binom{n}{k} \frac{k!k}{n^k} = n, \tag{1}$$

$$B \equiv \sum_{j=0}^{n} 2^{n-k-2j} \binom{n}{j}\binom{n-j}{j+k} = \binom{2n}{n+k}, \quad 0 \le k \le n. \tag{2}$$

Here we deduce the hypergeometric version of (1) and (2); in fact:

$$A = \sum_{k=0}^{\infty} t_k, \quad t_k = \frac{(n-1)!(k+1)}{(n-k-1)!n^k} \quad \therefore \quad \frac{t_{k+1}}{t_k} = \frac{(k+2)(k+1-n)}{k+1}\left(-\frac{1}{n}\right), \tag{3}$$

Hence [2-5] from (1) and (3) we obtain following special value for the hypergeometric function ${}_2F_0$ [6-9]:

$${}_2F_0\left(2, 1-n; -\frac{1}{n}\right) = n, \quad n \ge 1. \tag{4}$$

Similarly:

$$B \equiv 2^{n-k}\binom{n}{k}\sum_{r=0}^{\infty} t_r, \quad t_r = \frac{\binom{n}{r}\binom{n-r}{r+k}}{2^{2r}\binom{n}{k}} \quad \therefore \quad \frac{t_{r+1}}{t_r} = \frac{\left(r+\frac{k-n}{2}\right)\left(r+\frac{k-n+1}{2}\right)}{(r+k+1)(r+1)}, \tag{5}$$

Thus [2-5] from (2) and (5) we deduce an interesting value for ${}_2F_1$ [6-9]:

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$$ {}_2F_1\left(\frac{k-n}{2},\frac{k-n+1}{2};k+1;1\right)=\frac{\binom{2}{n+k}}{2^{n-k}\binom{n}{k}},\quad 0\le k\le n. \tag{6}$$

In [10] is the property:

$$C\equiv 2\sum_{k=1}^{n}\frac{H_k}{k+1}=\sum_{k=1}^{n}\frac{H_k}{n-k+1},\quad n\ge 1, \tag{7}$$

Involving harmonic numbers [11-13]; now we shall give a proof of (7), in fact, we know the following relation [14, 15]:

$$2\frac{H_k}{k+1}=\sum_{r=1}^{k}\frac{1}{r(k-r+1)}, \tag{8}$$

Therefore:

$$C=\sum_{r=1}^{n}\frac{1}{r}\sum_{k=r}^{n}\frac{1}{k-r+1}=\sum_{r=1}^{n}\frac{H_{n-r+1}}{r}=\sum_{j=1}^{n}\frac{H_j}{n-j+1},$$

In according with (7), q.e.d.

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