Approximate Solutions of Boundary-Value Problems in Multidimensional Cases

Zhanysova Aray Boshanovna, 2Nurkasymova Saule Nurkasymovna, 3Tuyakbaeva Sarah Boshanovna, 4Yermekova Zhadyra Kerimbaevna and 5Akzhigitov Erbulat Azhimovich

1Ph.D., West Kazakhstan State University named after M.Utemisov, Teacher of Physics and Mathematics, "Physics and Mathematics" department
2Doctor of Education, Professor of Eurasian National University named after L.N. Gumilev
3Department of the History of Kazakhstan and Social and Humanitarian Disciplines, Kazakh University of Technology and Business, Faculty of Economics and Technology
4Eurasian National University named after L.N. Gumilev
5Department of Higher Mathematics, Kazakh Agrotechnical University named after S.Seifullin

Abstract: An approximate solution of nonlinear boundary-value problems in non-standard areas is found in the following manner:

- Considering an equation and boundary conditions (in standard areas) for which the theorems on the smoothness, on the integral expression of a smooth function and on a solution continuation are correct.
- The solution obtained is substituted into the original equation and so integral equation is obtained.
- The initial problem is replaced by the variational problem of finding the functional implementing the minimum.

This approach is based on three theorems of the theory of embeddings and boundary-value problems:

- Smoothness of problems’ solutions,
- Theorems of embedding and continuation with preserving of smoothness class,
- Representation theorems.

The convergence degree of the proposed iterative method, based on the generation of minimizing sequence, which converges to the original problem, does not depend on a small parameter, which arises under the numerical implementation of difference schemes and fictitious domain method. These questions were open to non-linear operators. The proposed method can be treated as a new version of the fictitious domain method.

Key words: \(\Delta\)-Laplace operator \(\mathbb{R}^2\) with smooth boundary \(\partial\Omega\), \(Q\) bounded domain, defined space \(W^2_2(\Omega)\), integral operator, method of stability.

INRODUCTION

This work offers one of approaches to approximate solution of linear equations. The iteration process convergence is proved. As an example we considered a partial derivatives equation with a biharmonic operator, governing plate’s curves [1].

Let \(Q\) be a bounded domain, contained in \(\mathbb{R}^2\) with smooth boundary \(\partial\Omega\).

Let’s study the following problem

\[
Lu = -\Delta u + q(x)u = f(x)
\]

where \(x = (x_1,x_2)\), \(f(x) \in L^2(\Omega)\)

\[
\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}
\]

Let’s introduce an operator \(A_v\), represented in the following manner [2, 3].

\[
A_vu = v,
\]

\(A_v\)-Laplace operator, defined in the space \(W^2_2(\Omega)\).

After such replacement the equation (1) can be expressed as follows:

\[
u|_{\partial\Omega} = 0
\]
\[ Mv = v + q(x)A_0^{-1}v = f(x), \]  \tag{1'}

where \( x \in \mathbb{R}^2 \).

After finding \( v \) out of this integral equation, \( \theta \) is calculated by formula:

\[ u = A_0^{-1}v. \]

Let us denote

\[ J(\omega) = \int_{\Omega} [\omega + q(x)(A_0^{-1}\omega)(x) - f(x)]^2 dx, \]  \tag{3}

where an integral is understood as a double integral due to \( \Omega \).

Note that the solution (1') is equivalent to the solution (3).

Let us assume that an equation for any right-hand member of \( f(x) \in L_2(\Omega) \) has only one solution \([4, 5, 7]\). Then out of the Banach theorem \([6]\).

\[ \left\| M^{-1} \right\| = \left\| M^{*^{-1}} \right\| \leq c \times \infty, \]  \tag{4}

where

\[ \begin{align*}
M &= E + q(x)A^{-1}_0 \\
\text{We set} \quad \|M^n\|_R > \varepsilon \omega_{n+1} = \omega_n - E\omega. \quad \text{Then} \\
J(\omega_n) &= J(\omega_n - \varepsilon \omega) = \int_{\Omega} (\omega_n - \varepsilon \omega + q(x)A_0^{-1}(\omega_n - \varepsilon \omega) - f(x))^2 dx = \\
&= \int_{\Omega} (\omega_n + q(x)A_0^{-1}\omega_n + f(x))^2 dx - \\
&\quad -2 \int_{\Omega} (\omega_n + q(x)A_0^{-1}\omega_n - f(x))(\varepsilon \omega + q(x)A_0^{-1}(\varepsilon \omega))dx + \\
&\quad + \varepsilon^2 (\omega + q(x)A_0^{-1}(\omega))^2 dx = J(\omega_n) - 2\varepsilon \langle M\omega_n - f, M\omega \rangle + \varepsilon^2 \|M\omega\|^2 = \\
&= J(\omega_n) - 2\varepsilon \langle M\omega_n - f, M\omega \rangle + \varepsilon^2 \|M\omega\|^2.
\end{align*} \]

We used a condition that the integral operator \( A_0 \) is linear \([8]\). Let us choose \( \omega = M^*(M\omega_n - f) \). Then

\[ J(\omega_{n+1}) = J(\omega_n) - 2\varepsilon \|\omega\|^2 + \varepsilon^2 \|M\omega\|^2 \leq J(\omega_n) - [2\varepsilon - \|M\|^2 \varepsilon^2] \|\omega\|^2. \]  \tag{6}

Pursuant to the assumption (4) we have

\[ \sqrt{J(\omega_n)} = \|M\omega_n - f\| = \left\| M^{*^{-1}} M^* (M\omega_n - f) \right\| = \| M^{*^{-1}} \omega \| < c \|\omega\|. \]
Therefore from (6) it arises that
\[ J(\omega_{n+1}) \leq J(\omega_n) - \left[ 2\varepsilon - \|M\|^2 \varepsilon^2 \right] J(\omega_n) / c^2 = J(\omega_n) \left[ 1 - \left( 2\varepsilon - \|M\|^2 \varepsilon^2 / c^2 \right) \right]. \]

Let us choose \( \varepsilon = d^{-2} \leq \|M\|^2 \). Then we shall get
\[ J(\omega_{n+1}) \leq J(\omega_n) \left( 1 - \frac{1}{d^2 c^2} \right). \] (7)

Then from the remainder \( \omega_{n+1} - \omega_n \) we shall get
\[ \|\omega_{n+1} - \omega_n\| = \varepsilon \|M * (M \omega_n - f)\| \leq \varepsilon \|M\| \sqrt{J(\omega_n)} = \sqrt{J(\omega_n)} \]

The inequation obtained and (7) give the following result:

**Theorem 1:** Let it be that for any \( f(x) \in L_2(\Omega) \) the problem (1) has only one solution \( u \in W^1_2(\Omega) \), and the operator \( M \), defined due to (1'), satisfies the conditions \( \|u\| \leq M \|M \omega_n - f\| \) and the inequation \( cd > 1 \) is correct. Then for any \( \omega_n \in L_2(\Omega) \) the sequence \( \{\omega_n\} \), defined by formulas
\[ \omega_{n+1} = \omega_n - d^{-2} M * (M \omega_n - f) \]

converges to the solution of \( \Omega \) equation (1') and also following estimations are fulfilled:
\[ J(\omega_n) \leq \Theta J(\omega_0), \]
\[ \|\omega_{n+1} - \omega_n\| \leq c \eta^{n/2} \sqrt{J(\omega_n)}, \]

where \( \eta = 1 - \frac{1}{d^2 c^2} \). Whereas the function \( u = A^{-1} \omega \) shall be the solution to the problem (1.31).

The result of this theorem gives the problem (1) solution algorithm [9]. And the sequence \( \omega \) converges to the solution of the problem (1') at rate of geometrical progression. Because during the transition from (1) to (1') we put \( -\Delta u = v \), then \( u = (\Delta)^{-1} \omega \), converges to the solution (1.31) at rate of geometrical progression in the Sobolev’s metric \( 0 \in W^1_2(\Omega) \).

The method is stable as regard to computational errors.
Here we can get computation similar to this in (1).

In (3), considering \( \omega \) to be dependent on the parameter \( t \in (0, \infty) \) let us differentiate \( J(\omega) \) with respect to \( t \):
\[ J'_t(\omega) = 2 \left\{ \omega, M * (M \omega - f) \right\} \]

Let us pick \( \Omega \) from the equation
\[ \omega = - \frac{1}{2} M * (M \omega - f). \] (9)

Then from (8) it follows
\[ J'_t(\omega) = - \|\omega\|^2 = - \|M * (M \omega - f)\|^2. \]

Because of this equation and because of boundedness \( M *^{-1} \) let us define \( J(\omega) \) with the help of the operator \( M \), therefore: \( -J'_t(\omega) \geq c^{-2} J(\omega) \).
From this inequation, solving ordinary differential equation, we get, using (9), that $J(\Omega)$ exponentially decreases. For solving (9) we have

$$\omega(t) = e^{-B(t-t_0)}\omega(t_0) + B^{-1}(e^{-B(t-t_0)})\tilde{f}$$  \hspace{1cm} (10)

$t\geq t_0\geq 0$, where $B = M*\omega/2$, $\tilde{f} = M*f$.

Then, as we have seen above, we shall get an exponential convergence $\omega(t)$ to boundary value with $t\to\infty$. But we do not need the solution (9) for all values $t$ (this would be a difficult problem), we need only a boundary value, to which $\omega(t)$ converges with $t\to\infty$, i.e. we need the element which is a stable position for the solution (9). Transition to the boundary value in (10) shall lead us to the original problem, because

$$\lim_{t\to\infty} \omega(t) = B^{-1}\tilde{f}$$

Therefore the formula (10) has to be replaced by another one, which can be found easily and has a boundary value which is the same with the vector-function $\Omega(t)$ in infinitude.

Let us consider a particular case of free oscillations of plate located on an elastic foundation and hinge-supported along the contour; the case is described by a differential bielliptic equation with homogeneous boundary conditions:

$$Lu = \Delta(\Delta u) + q(x)u = f(x)$$

$$u\bigg|_{x_1=0} = 0, u\bigg|_{x_2=1} = 0$$

$$\frac{\partial^2 u}{\partial x_1^2}\bigg|_{x_1=0} = 0, \frac{\partial^2 u}{\partial x_1^2}\bigg|_{x_1=1} = 0, \frac{\partial^2 u}{\partial x_2^2}\bigg|_{x_2=0} = 0, \frac{\partial^2 u}{\partial x_2^2}\bigg|_{x_2=1} = 0,$$

where $x = (x_1,x_2)$, $f(x)$- external force, $f(x)\in L_4(\Omega)\Omega = (0,1)\times(0,1)$- open bounded domain, $q(x)u(x)$ - reaction pressure of elastic foundation and $q(x)\geq 0$.

Let us introduce an operator $A_0$, which is represented as follows:

$$A_0u \equiv \Delta(\Delta u) = v,$$  \hspace{1cm} (11)

where $A_0$ is a biharmonic operator, defined in the space $\mathbb{H}_2^4(\Omega)$.

With above mentioned boundary conditions this problem is solved in the following way:

$$u = A_0^{-1}v = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4}{(\pi n)^2 + (\pi m)^2} \times \sin \pi n x_1 \cdot \sin \pi m x_2 \cdot \int_{0}^{1} \int_{0}^{1} v(\xi,\eta) \cdot \sin \pi n \xi \cdot \sin \pi m \eta \cdot d\xi d\eta$$  \hspace{1cm} (12)

Substituting (1.12) into the equation (1.11) we shall get:

$$Mv = v + q(x) \cdot A_0^{-1}v = f(x)$$  \hspace{1cm} (13)

After we find out $v$ from this integral equation (1.13), $u$ is calculated by the formula:

$$u = A_0^{-1}v.$$

Now let us concern the problem we at issue.
Let us put
\[ J(\omega) = \int_{\Omega} \omega + q(x)(A^{-1}_0 \omega)(x) - f(x) \, dx \]  
(14)

where an integral is understood as a double integral in the area $\Omega$.

Note that the solution (13) is equivalent to the solution (14).

Let us assume that the equation for any right-hand member of $f(x) \in L_2(\Omega)$ has only one solution.

We construct an iteration process.

Let
\[ \omega_{k+1} = \omega_k - \varepsilon \omega, \quad \|M\| > \varepsilon. \]

**Results of a Numerical Experiment:** As an example let us consider the following problem:

\[ \Delta(\Delta u) + x_1 x_2 u = (4\pi^4 + x_1 x_2) \sin \pi x_1 \sin \pi x_2 \]

\[ u \bigg|_{x_1=0} = u \bigg|_{x_1=1} = 0 \quad u \bigg|_{x_2=0} = u \bigg|_{x_2=1} = 0 \]

\[ \frac{\partial^2 u}{\partial x_1^2} \bigg|_{x_1=0} = 0, \quad \frac{\partial^2 u}{\partial x_1^2} \bigg|_{x_1=1} = 0, \quad \frac{\partial^2 u}{\partial x_2^2} \bigg|_{x_2=0} = 0, \quad \frac{\partial^2 u}{\partial x_2^2} \bigg|_{x_2=1} = 0. \]

For this problem the equation (1.13) is as follows:

\[ Mv = v + x_1 x_2 A^{-1}_0 v = (4\pi^4 + x_1 x_2) \sin \pi x_1 \sin \pi x_2, \]

We find out the meaning of a functional for the element $\omega_{n+1}$ according to the following scheme:

\[ J(\omega_{k+1}) = J(\omega_k - \varepsilon \omega) = \]

\[ \int_{\Omega} (\omega_k - \varepsilon \omega + x_1 x_2 A^{-1}_0 (\omega_k - \varepsilon \omega) - 4\pi^4 x_1 x_2 \sin \pi x_1 \sin \pi x_2) \, dx_1 dx_2 = \]

\[ \int_{\Omega} (\omega_k + x_1 x_2 A^{-1}_0 \omega_k - 4\pi^4 x_1 x_2 \sin \pi x_1 \sin \pi x_2)^2 \, dx_1 dx_2 - \]

\[ 2 \int_{\Omega} (\omega_k + x_1 x_2 A^{-1}_0 \omega_k - 4\pi^4 x_1 x_2 \sin \pi x_1 \sin \pi x_2) \times \]

\[ \left( \varepsilon \omega + x_1 x_2 A^{-1}_0 (\varepsilon \omega) \right) \, dx_1 dx_2 + \]

\[ \varepsilon^2 \int_{\Omega} (\omega_k + x_1 x_2 A^{-1}_0 (\omega_k)) \, dx_1 dx_2 = J(\omega_k) - 2 \varepsilon \langle M \omega_k - f, \omega \rangle + \varepsilon^2 \|M \omega\|^2 = \]

\[ J(\omega_k) - 2 \varepsilon \langle M^* (M \omega_k - f), \omega \rangle + \varepsilon^2 \|M \omega\|^2, \]

where $\omega_{k+1} = \omega_k - \varepsilon \omega$, $\omega = M^* (M \omega_k - f)$, $M^*$ - adjoint operator of the operator $M$, i.e.

\[ \langle Mv, w \rangle = \left\langle v, M^* w \right\rangle \]
and the symbol \( \int \int \) means a square double integral: \( \Omega = \{0 \leq \eta \leq 1, 0 \leq x_2 \leq 1\} : \\
\int \Omega (\cdot)dx_1dx_2 = \int \int \Omega (\cdot)dx_1dx_2.

In this case the operator \( M^* \), which is adjoint to the operator \( M \) and a residual of a kind \( M\omega_k - f \) are as follows, accordingly:

\[
M^*(\cdot) = E + 4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin \pi n \xi \cdot \sin \pi m \eta}{((\pi n)^2 + (\pi m)^2)^2} \int \int \Omega \frac{1}{0} x_1 \cdot x_2 \cdot \sin \pi nx_1 \cdot \sin \pi nx_2 \cdot dx_1dx_2,
\]

(15)

and a residual \( M\omega - f \) is expressed by the formula

\[
M\omega - f = \omega_k + 4x_1x_2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin \pi nx_1 \cdot \sin \pi mx_2}{((\pi n)^2 + (\pi m)^2)^2} \times \\
\int \int \Omega \frac{1}{0} \frac{\omega_k}{0} \frac{\sin \pi n \xi \cdot \sin \pi m \eta \cdot d\xi d\eta}{d\eta} - (4\pi^4 + x_1x_2) \times \\
\times \sin x_1\pi \cdot \sin x_2\pi \cdot k \in N.
\]

Then, in view of (15) and (16), we shall get

\[
\omega = M^*(M\omega_k - f) = \omega_k + 4x_1x_2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\sin \pi nx_1 \cdot \sin \pi mx_2}{((\pi n)^2 + (\pi m)^2)^2} \times \\
\int \int \Omega \frac{1}{0} \frac{\omega_k}{0} \frac{\sin \pi n \xi \cdot \sin \pi m \eta \cdot d\xi d\eta}{d\eta} - (4\pi^4 + x_1x_2) \times \\
\times \int \int \Omega \frac{1}{0} \frac{\omega_k}{0} \frac{\sin \pi n \xi \cdot \sin \pi m \eta \cdot d\xi d\eta}{d\eta} - \\
-(4\pi^4 + x_1x_2) \cdot \sin x_1\pi \cdot \sin x_2\pi \cdot dx_1dx_2.
\]

Making an approximate calculation, let us assume \( n = \overline{1,N} \) and \( m = \overline{1,N} \) [10, 11]:

\[
\omega = M^*(M\omega_k - f) = \omega_k + 4x_1x_2 \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{\sin \pi nx_1 \cdot \sin \pi mx_2}{((\pi n)^2 + (\pi m)^2)^2} \times \\
\int \int \Omega \frac{1}{0} \frac{\omega_k}{0} \frac{\sin \pi n \xi \cdot \sin \pi m \eta \cdot d\xi d\eta}{d\eta} - (4\pi^4 + x_1x_2) \cdot \sin x_1\pi \cdot \sin x_2\pi + \\
\sum_{n=1}^{N}\sum_{m=1}^{N} \frac{\sin \pi n \xi \cdot \sin \pi m \eta \cdot d\xi d\eta}{d\eta} - (4\pi^4 + x_1x_2) \cdot \sin x_1\pi \cdot \sin x_2\pi \times \\
\times \sin x_1\pi \cdot \sin x_2\pi \cdot dx_1dx_2.
\]
Table 1: Exact solution of a bielliptic equation

<table>
<thead>
<tr>
<th>x_1</th>
<th>x_2</th>
<th>x_3</th>
<th>x_4</th>
<th>x_5</th>
<th>x_6</th>
<th>x_7</th>
<th>x_8</th>
<th>x_9</th>
<th>x_10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.05</td>
<td>0.00</td>
<td>0.0483</td>
<td>0.0919</td>
<td>0.1266</td>
<td>0.1488</td>
<td>0.1564</td>
<td>0.1266</td>
<td>0.0919</td>
<td>0.0483</td>
</tr>
<tr>
<td>0.10</td>
<td>0.00</td>
<td>0.0955</td>
<td>0.1816</td>
<td>0.2500</td>
<td>0.2939</td>
<td>0.3090</td>
<td>0.2939</td>
<td>0.2500</td>
<td>0.1816</td>
</tr>
<tr>
<td>0.15</td>
<td>0.00</td>
<td>0.1403</td>
<td>0.2668</td>
<td>0.3673</td>
<td>0.4318</td>
<td>0.4540</td>
<td>0.4318</td>
<td>0.3673</td>
<td>0.2668</td>
</tr>
<tr>
<td>0.20</td>
<td>0.00</td>
<td>0.1816</td>
<td>0.3455</td>
<td>0.4755</td>
<td>0.5590</td>
<td>0.5878</td>
<td>0.5590</td>
<td>0.4755</td>
<td>0.3455</td>
</tr>
<tr>
<td>0.25</td>
<td>0.00</td>
<td>0.2185</td>
<td>0.4156</td>
<td>0.5721</td>
<td>0.6725</td>
<td>0.7071</td>
<td>0.6725</td>
<td>0.5721</td>
<td>0.4156</td>
</tr>
<tr>
<td>0.30</td>
<td>0.00</td>
<td>0.2500</td>
<td>0.4755</td>
<td>0.6645</td>
<td>0.7609</td>
<td>0.7969</td>
<td>0.6645</td>
<td>0.4755</td>
<td>0.3000</td>
</tr>
<tr>
<td>0.35</td>
<td>0.00</td>
<td>0.2753</td>
<td>0.5237</td>
<td>0.7208</td>
<td>0.8474</td>
<td>0.8910</td>
<td>0.8474</td>
<td>0.7208</td>
<td>0.5237</td>
</tr>
<tr>
<td>0.40</td>
<td>0.00</td>
<td>0.2939</td>
<td>0.5590</td>
<td>0.7694</td>
<td>0.9045</td>
<td>0.9511</td>
<td>0.9045</td>
<td>0.7694</td>
<td>0.5590</td>
</tr>
<tr>
<td>0.45</td>
<td>0.00</td>
<td>0.3052</td>
<td>0.5805</td>
<td>0.7991</td>
<td>0.9393</td>
<td>0.9877</td>
<td>0.9393</td>
<td>0.7991</td>
<td>0.5805</td>
</tr>
<tr>
<td>0.50</td>
<td>0.00</td>
<td>0.3090</td>
<td>0.5878</td>
<td>0.8090</td>
<td>0.9511</td>
<td>1.0000</td>
<td>0.8090</td>
<td>0.9511</td>
<td>0.5878</td>
</tr>
</tbody>
</table>

The functional \( J \) is of the form of

\[
J(x_1, x_2) = \int_0^1 \int_0^1 M(x_1, x_2) \cdot \sin x_1 \cdot \sin x_2 \cdot d\xi d\eta - (4\pi^4 + x_1 x_2) \cdot \sin x_1 \cdot \sin x_2 \cdot \pi^2 \cdot dx_1 dx_2.
\]

The numerical work is done in the same way as in the first example. Below you can find tables of numerical work and graphic illustrations to them.

Each table (tab. 1) shows numerical values of exact approximate solutions in correspondent points of a plane \((x_1, x_2)\), \(i = 0, 2m, j = 0, 2n\). The table 1 shows exact numerical solutions for the example analyzed, an exact analytical solution to which is of the form

\[ u(x_1, x_2) = \sin x_1 \cdot \sin x_2. \]

REFERENCES


