An Optimization Approach for Highway Vertical Alignment Using the Earthwork Balance Condition

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Abstract: Design of highway vertical alignments mainly affects road construction cost, safety and traffic operational performance. Vertical alignment optimization includes minimizing of the total amount of earthwork in addition to fulfilling the design criteria. Traditionally, determination of optimum vertical alignment yields to trial and error process and consumes considerable time even so in the case of one vertical curve. This could be due to the absence of mathematical relationships among the design variables. In the last few decades, there are many approaches for vertical alignment optimization using linear programming and genetic algorithms. However, most of these approaches assumed that the vertical curves are located at equal distances. This assumption will not lead to the absolute least earthwork amount. On the other hand, most of these approaches did not consider the design criteria concerning the vertical curves such as the rate of curvature as well as the minimum grades. In this paper, a proposed approach for optimizing highway vertical alignment using the condition of earthwork balance was presented. The earthwork balance condition leads to equal cut and fill quantities. In addition, it leads to less amount of earthwork. The proposed approach determines the best location and level for the desired vertical curve achieving as possible as the earthwork balance, less earthwork amount. In addition, it takes into consideration the design criteria for grades and vertical curve including the rate of curvature as well as the minimum grades. The earthwork balance condition is considered a new relationship between the design variables. The optimization process will search for only one design variable and consequently the consumed time will be less than that of the traditional approach. The proposed approach was tested using a numerical example. The results ensured that the proposed approach leads to minimal earthwork quantity close to the least earthwork quantity in less time. The proposed approach in this paper considers only the case of one vertical curve; however, it can be expanded to the case of several vertical curves in the future work.

Keywords: Design criteria · Earthwork balance · Vertical alignment elements

INTRODUCTION

Design of highways vertical alignments is a very important task. It affects mainly the cost of road construction, road safety and the operational performance of highways. The main constraints in the vertical design are the design criteria such as the grades and rate of curvature. In fact, there are several designs achieving all design criteria. The best design should consider the earthwork objective which varies from case to another. Some of earthwork objectives are related to the earthwork amount such as the least cut, least fill and least total earthwork. On the other hand, the earthwork objectives may be related to the earthwork cost which depends on the earthwork amount as well as fill and cut locations.

Traditionally, the optimum design is obtained by trial and error process and needs much effort and time especially in the case of multi vertical curves where the locations and levels for all vertical curves are considered independent unknown variables. Even so in the case of one vertical curve, there is no relationship between the location and level of that vertical curve. Therefore, the search process will be for each location and level independently. In the case of one vertical curve, the
search process can be for determining the best grades which achieve the design criteria and least earthwork. Therefore, a two-loop computer program can be used to assess all available values for the second grade at certain available value for the first grade and record the earthwork amount for each pair of grades. The optimum design is the design corresponding to the minimum earthwork amount. In the last few decades, there are many approaches for vertical alignment optimization. Easa [1] used the linear programming to select the roadway grades that minimize the cost of earthwork and satisfy the maximum grade criterion using two separated stages; one for optimizing the grade selection and the other for optimizing the cost of earthwork.

In contrast, Moreb [2] approach combines the grade selection stage and the earthwork allocation stage in a single linear programming problem. Easa [1] and Moreb [2] modeled the roadway profile as piecewise linear segments. Therefore, sharp connectivity is created at the point where linear segments meet. Moreb [3] treated the sharp connectivity in the same linear programming by transformed the vertical alignment into some successive spline curves. However, this approach did not take into consideration the rate of curvature criterion. In another approach, Wang et al. [4] divided the road length into equal parts and used the genetic algorithms to determine the best level of each vertical point of intersection (VPI). Then they reduced the number of VPI’s according to the deflection angles at each VPI. Then they used the genetic algorithm again to get the best levels to achieve the design criteria.

On the other hand, Goktepe et al. [5] applied the genetic algorithm for optimizing vertical alignment taking into consideration the soil type. The genetic algorithm approach can lead to an optimal design if the road length is divided into large number of equal parts. Unfortunately, it did not consider the rate of curvature criterion which controls the part length. Generally, the earthwork amount cannot be derived as a function of the main design parameters due to the irregular shape of the ground line. Therefore, the optimum design can’t be obtained mathematically and the trial and error process will be an applicable procedure [5]. The optimization process can use the traditional direct search, linear programming, or genetic algorithms. The traditional approach is easier to be understood and programmed than linear programming and genetic algorithms approaches. In addition, the non-traditional approaches will stop at any optimal solution, if exists, not at the absolute optimum solution. In case of multi optimal solutions, the non-traditional approaches will stop at the optimal solution related to the initial seed and the consumed time can’t be predicted. In contrast, the traditional approach reaches the optimal solution with predictable time. The main disadvantage of the traditional approach is the considerable consumed time especially if the optimization process includes multiple independent unknowns.

In this paper, a proposed approach for optimizing the vertical alignment, containing one vertical curve, is presented. This approach can be generalized in the future researches for the multiple vertical curve cases. The proposed approach tries to determine the best two grades achieving less earthwork quantity and taking into consideration all important design criteria using the tradition direct search. This approach depends on the relationship between grades under the earthwork balance condition. It’s found from this paper that the earthwork balance condition leads to less earthwork quantity close to the absolute least earthwork quantity.

This paper is divided into five sections. After this introduction, the second section introduces the basic definitions that describe the design criteria fulfilled in this paper. New mathematical relationships concerning the earthwork optimization are presented in the third section. The proposed approach is explained in the fourth section along with some successive steps. A numerical example, to explain the proposed approach, is presented in the fifth section.

Basic Definitions: In this section, some basic definitions will be stated to highlight the important technical terms concerning the vertical alignment elements, the important design criteria which will be fulfilled in the proposed approach.

Vertical Alignment Elements: Vertical alignment mainly consists of two elements; grades and vertical curves. The grades are straight lines while the vertical curves are parabolic curves between the successive grades. The objective of vertical curves is to enable vehicles to move smoothly and safely between grades. Each element is determined based on its design criteria. In addition, the grade elements are determined to minimize as possible as the earthwork needed to transform the ground profile to the design profile. The major design criteria will be stated in the following articles.

Allowable Grades: Based on the design speed and the traffic composition, the grades should be not great than certain value called the maximum grade \(G_{\max}\). For rain
consideration, the grades should be not less than certain value called the minimum grade (G_{min}). The road specifications, such as the American Association of State Highway and Transportation Officials (AASHTO) [6], state both the maximum and minimum grades.

**Rate of Curvature:** Rate of curvature (k) is the length of vertical curve per percent algebraic difference in the intersecting grades. It can be expressed as follows:

\[
k = \frac{L}{A}
\]

(1)

where:

L is the vertical curve length by meter.

A is the algebraic difference in the grades before and after the vertical curve.

The great value of (k) leads to more safety. However, it leads to greater length of vertical curve, which causes extra cost in the construction process. The design specifications such as AASHTO [6] states that the minimum value of k is based on the initial speed and the vertical curve type. There are two types of the vertical curve; crest and sag. In this paper, the minimum value of (k) for crest vertical curve will be denoted \((k)_c\) and for sag vertical curve will be denoted \((k)_s\).

**New Mathematical Relationships:** In this section, new mathematical relationships connecting the important design criteria and the terms of earthwork will be presented. These relationships will be used basically in the proposed approach.

**Outside Area:** Outside area for the vertical curve \((A_{out})\) is bounded by the vertical curve and its two tangents. Fig.1 indicates the outside area in the crest and sag cases. Outside area for the crest curve and the sag curve can be denoted \((A_{oc})\) and \((A_{os})\), respectively. By integration of the equation of vertical curve, the following equations were derived which determine the outside area in terms of the two grades as well as the rate of curvature. Equation 2 computes outside area for the crest curve case while equation 3 computes outside area for the crest curve case.

\[
A_{oc} = \frac{(G_1 - G_2)^2}{2400} K_c^2
\]

(2)

\[
A_{os} = \frac{(G_2 - G_1)^2}{2400} K_s^2
\]

(3)

where:

\(G_1\) and \(G_2\) are the grade percent of the first and second tangents, respectively.

**Area under Design Line:** Area under design line \((A_d)\) is the area bounded by the design line and the datum, which should not intersect the design line or ground line. Area under design line depends on the number of vertical curves, their locations and the grades between them. In this paper, the design line will have one vertical curve. Fig. 2 indicates area under design line in the case of one crest vertical curve. The station and level of VPI can be denoted by \(X_c\) and \(Y_c\), respectively. The following equations were derived to determine the location and level of VPI in terms of the two grades as well as the coordinates of control points regardless the curve type.

\[
X_v = \frac{100(Y_e - Y_0) + G_1X_0 - G_2X_e}{(G_1 - G_2)}
\]

(4)

\[
Y_v = \frac{Y_0 + G_1(X_v - X_0)}{100}
\]

(5)

where:

\(X_0\) and \(Y_0\) are the station and level of the control point at the road start respectively.

Fig. 1: Outside area for crest and sag curves.
Fig. 2: Area under design line having one crest vertical curve.

Fig. 3: Earthwork balance condition.

X₀ and Y₀ are the station and level of the control point at the road end, respectively. G₁ and G₂, as defined above, are the grades before and after the vertical curve.

In addition, the area under design line in the crest case which can be denoted (A_c) was calculated and found to be as follows:

\[
A_{dc} = \frac{1}{2}((X_0 - X_c)(Y_0 + Y_c) + (X_c - X_e)(Y_c + Y_e)) - \frac{(G_1 - G_2)^2}{2400} K_c^2
\]  

Similarly, the area under design line in the sag case which can be denoted (A_s) was calculated and found to be as follows:

\[
A_{ds} = \frac{1}{2}((X_0 - X_s)(Y_0 + Y_s) + (X_s - X_e)(Y_s + Y_e)) - \frac{(G_1 - G_2)^2}{2400} K_s^2
\]  

**Balanced Earthwork:** Balanced earthwork means that the total cut quantity and total fill quantity are the same. This case may be useful for several cases such as when the soil of cut is valid for fill process, in this case there is no transportation cost of soil from or to outside the road site. As will be discussed latter, the balanced case is useful to minimize the total earthwork quantity and consequently the construction cost regardless the soil type. Therefore, if the soil of cut is not valid for fill process, the transportation cost of soil, will be minimal in the balanced case. Fig. 3 indicates the ground profile, design line, cut zones, fill zones and common area under both ground and design lines which can be denoted (A_u). The following equations can be easily stated:

\[
A_e = A_u + \Sigma C
\]  

\[
A_i = A_u + \Sigma F
\]  

where:

- \(A_e\) is the area under the ground line for the whole road length.
- \(A_u\) is the area under design line for the whole road length.
- \(A_i\) is the area under both ground and design lines for the whole road length.
- \(\Sigma C\) is the total cut amount for the whole road length.
- \(\Sigma F\) is the total fill amount for the whole road length.

Referring to equations 8 and 9, it is clear that if \(A_e = A_i\) then the total cut will be equal to the total fill and then the balanced case will be occur i.e. the earthwork balance occurs when the area under design line (\(A_u\)) equal to the area under the ground line(\(A_e\)) for the whole road length.
**Unbalanced Earthwork:** Unbalanced earthwork case occurs when the design line can’t achieve the area under the ground line due to the design criteria or the position of control points. Therefore, the resulted earthwork will be borrow or excess. Fig. 4 indicates the excessive earthwork case. It is clear that the position of control points, $P_1$ and $P_2$ and the low value of maximum grades $G_{\text{max}}$ are obstacles to achieve the balanced earthwork condition. In the unbalanced case, the optimal design line is the design line achieves the design criteria and the least borrow or excess. The design line in this case will have one vertical curve with maximum grades.

**Optimum Vertical Alignment:** The optimum vertical alignment in the case of one vertical curve means the determination of the curve type and the grades which achieve the design criteria and least amount of earthwork quantity. Determination of the vertical curve type depends on the ground profile as well as the coordinates of control points. In other words, the curve type can be determined by the area difference ($\Delta A$) between area under the ground line $A_g$ and the basic area $A_b$, where:

\[
\Delta A = A_g - A_b
\]

\[
A_g = \frac{1}{2} \sum (X_i - X_{i-1})(Y_i - Y_{i-1})
\]

where:

- $X_i$ and $Y_i$ are the station and level of $i^{th}$ point in the ground profile.
- $A_g$ is the area bounded by the datum and the direct line connecting the two control points; $(X_a, Y_a)$ and $(X_e, Y_e)$. The basic area is defined as follows:

\[
A_b = \frac{1}{2} (X_e - X_0)(Y_e + Y_0)
\]

To minimize the earthwork quantity, if $\Delta A > 0$ then a crest vertical curve is required while if $\Delta A < 0$ then a sag vertical curve is required. In addition, the grades should be determined to minimize the difference between area under the ground line and area under design line. If there allowable grades can achieve the balance condition, the earthwork balance case will be occurred otherwise the unbalance case will be occurred. At the unbalanced case, if $\Delta A > 0$, then the optimal design will have one crest vertical curve and the first and second grades should be $G_{\text{max}}$ and $-G_{\text{max}}$ respectively. Substituting by these grades in equations 4 and 5, $X_0$ and $Y_0$ can be determined. Similarly, if $\Delta A < 0$, then the optimal design will have one sag vertical curve and the first and second grades should be $-G_{\text{max}}$ and $G_{\text{max}}$ respectively. Substituting by these grades in equations 4 and 5, $X_e$ and $Y_e$ can be determined.

At the balanced case, the left side in both equations 6 or 7, according to the curve type, should equal to the area under the ground line $A_g$. Substituting by $X_c$, $Y_c$ and $A_b$ defined in equations 4, 5 and 13 respectively, in equations 6 or 7 leads to the following equation:

\[
2400\Delta A \Delta G + 12 (100\Delta Y \cdot G_c \Delta X)(100\Delta Y \cdot -G_s \Delta X) + K^2 \Delta G = 0
\]

where:

- $\Delta G = G_1 \cdot G_2$, $G_1$ and $G_2$ are the grades before and after the vertical curve, respectively.
- $\Delta X = X_c \cdot X_0$, $X_e$ and $X_c$ are the stations at road start and road end, respectively.
- $\Delta Y = Y_c \cdot Y_0$, $Y_e$ and $Y_c$ are the control levels at road start and road end, respectively.
- $K = K_c$ in the crest case and $K = K_s$ in the sag case.

For certain value of $G_i$, the corresponding values of $G_2$ can be calculated by solving equation 13. In this case and due to the fourth degree of equation 13, $G_2$ will have four values; often two imaginary values and two real values. The best value of $G_i$ should be selected from the real values to achieve the design criteria. If the real values of $G_2$ don’t achieve the design criteria, then the value of $G_i$ is not applicable to achieve the balanced earthwork.
Steps of Executing the Proposed Approach: The main objective of earthwork optimization is to minimize the total amount of earthwork. In the case of one vertical curve, any design line is determined by only two parameters; \( X_v \) and \( Y_v \), which can be used to determine the remaining design parameters such as the first and second grades \( G_1 \) and \( G_2 \) as well as the curve length and levels. In other words, the two grades \( G_1 \) and \( G_2 \) can be considered the main design parameters which can be used to obtain the remaining parameters such as the location and level of vertical curve as well as the curve length. In the two procedures, the corresponding earthwork amount can be calculated using the ground line in addition to the design line. In the balanced earthwork case, the least earthwork quantity will be determined by examining all allowable values for the first and second grades taking into consideration both other design criteria and the balance condition. Equation 13 will be used to get the second grade corresponding to certain value of the first grade. The actual range of the first grade \([G_{1\text{min}}, G_{1\text{max}}]\) can be determined using equation 13 as will be explained below. For each value of the first grade, the earthwork amount can be calculated and the optimum design is that design leads to the least earthwork quantity. The following steps explain the proposed approach in details:

Earthwork Balance Condition:

- Compute the area under the ground line \( (A_g) \) using equation 11.
- Compute the basic area \( (A_b) \) under the direct line connecting the two control points which described in equation 12.
- Then compute the difference in the two areas \( \Delta A \) as defined in equation 10.
- If \( \Delta A > 0 \) then a crest curve is required to achieve the earthwork balance condition, therefore the following steps should be carried out:
  - Let \( G_1 = G_{\text{max}} \) and \( G_2 = -G_{\text{max}} \) then compute \( X_v \) and \( Y_v \) using equations 4, 5, and 6.
  - If \( A_{d1} > A_{d2} \) then the earthwork balance can be achieved, otherwise an excessive case will be obtained and the total excess equals \( A_{d1} - A_{d2} \).
- If \( \Delta A < 0 \) then a sag curve is required to achieve the earthwork balance condition, therefore the following steps should be carried out:
  - Let \( G_1 = -G_{\text{max}} \) and \( G_2 = G_{\text{max}} \) then compute \( X_v \) and \( Y_v \) using equations 4, 5, and 7.
  - If \( A_{d1} < A_{d2} \) then the earthwork balance can be achieved, otherwise a borrow case will be obtained and the total borrow equals \( A_{d1} - A_{d2} \).
- If the earthwork balance can be achieved then the following steps should be carried out to reach the best design line.

First Grade Actual Range: As mentioned above, the first and second grades are related by equation 13. In the crest case, the first and second grade ranges are \([G_{\text{max}}, G_{\text{max}}]\) and \([-G_{\text{max}}, -G_{\text{max}}]\) respectively due to the design criteria. By substituting with the upper and lower limit of the second grade in equation 13, the corresponding range of the first grade will be \([G_{1\text{min}}, G_{1\text{max}}]\). Therefore the first grade actual range, which can be denoted \([G_{1\text{min}}, G_{1\text{max}}]\), will be the common range between the two ranges \([G_{\text{max}}, G_{\text{max}}]\) and \([G_{1\text{min}}, G_{1\text{max}}]\). Similarly, in the sag case, the first and second grade ranges are \([-G_{\text{max}}, -G_{\text{max}}]\) and \([G_{\text{min}}, G_{\text{min}}]\) respectively due to the design criteria. By substituting with the upper and lower limit of the second grade in equation 13, the corresponding range of the first grade will be \([G_{1\text{min}}, G_{1\text{max}}]\). Therefore the first grade actual range, which is denoted \([G_{1\text{min}}, G_{1\text{max}}]\) will be the common range between the two ranges \([-G_{\text{max}}, -G_{\text{max}}]\) and \([G_{1\text{min}}, G_{1\text{max}}]\).

Least Earthwork Quantity: Each value of the first grade \( G_i \) in the actual range leads to a corresponding value of the second grade \( G_{1i} \) according to equation 13. The corresponding station and level of the vertical point of intersection of the vertical curve \( X_i \) and \( Y_i \) can be computed by equations 4 and 5. The corresponding earthwork quantity \( E_i \) can be computed as the total area bounded by the ground line and the design line which is described by \( G_{1i} \), \( G_{2i} \), \( X_i \), and \( Y_i \). Unfortunately, the relation between earthwork quantity \( E_i \) and the corresponding value of the first grade \( G_{1i} \) is nonlinear and may have more than one minimal value. Therefore, the least earthwork quantity should be determined by examining all values in the actual range of the first grade and computing the corresponding values of earthwork quantities and then comparing these values to select the least earthwork quantity \( E_{min} \) and the corresponding values of \( G_{1i} \), \( G_{2i} \), \( X_i \), and \( Y_i \). Practically, the actual range of the first grade should be divided into segments using a small value \( \Delta G \), such as 0.01%. The following example will validate the proposed approach. In addition, the absolute least earthwork quantity will be computed traditionally by examining each allowable pair of the first and second grades. In all cases the grade range will be divided every 0.01%. The resulted earthwork quantities will be compared as well as the consumed time. Matlab software is used to run the traditional and the proposed approach by writing an m-file for each technique.
Fig. 5 illustrates the ground line for a road of 350 m length. The control points at the start and end of road have the levels 15 and 17, respectively. The major design criteria are as follows:

- The maximum and minimum allowable grades are 3.5% and 0.5%, respectively.
- The rate of curvature for crest and sag curves are 11 and 18, respectively.

### Earthwork Balance Condition:

- \(A\) (area under the ground line to the datum) = 2500 m$^2$
- \(A\) (area under the straight line connection the two control points) = 2100 m$^2$
- \(\Delta A = 400\) m$^2$, then a crest curve is required.
- \(A_1\) (at \(G_1 = 3.5\) and \(G_2 = -3.5\)) = 3143 m$^2$
- \(A_2 > A_1\), then the balanced case can be achieved using crest vertical curve.

### First Grade Actual Range:

- \(G_{1\min}\) (at \(G_1 = -3.5\)) = 1.36
- \(G_{1\max}\) (at \(G_1 = 0.5\)) = 2.25
- \(G_{1\min}\) = 1.36 (max of 0.5 and 1.36)
- \(G_{1\max}\) = 2.25 (min of 3.5 and 2.25)

### Least Earthwork Quantity:

- The allowable values of the first and second grades start from 0.5% to 3.5% every 0.01%.
- The number of grade pairs including the first and second grades = 90301.
- For each grade pair, the remaining design parameters including the station and level of vertical point of intersection were calculated using equations 4 and 5.
- The vertical curve start and end as well as the earthwork quantity were calculated from the ground and design lines.
- The least earthwork quantity = 382 m$^2$ (fill = 229 and cut = 153 m$^2$)
- The least earthwork quantity occurs at \(G_1 = 3.5\), \(G_2 = -0.5\), \(X_1 = 93.750\) and \(Y_1 = 18.281\).
- The previous calculation consumed 4.84 minutes by the same processor.

This numerical example can indicate that the proposed approach achieves the earthwork balance. The earthwork quantity obtained is close to the absolute least earthwork by 1% tolerance. In addition, the proposed approach takes 10% of time consumed by the traditional approach using the same computer and software.

### Conclusions and Future Work:

A proposed approach for optimizing highway vertical alignment using the condition of earthwork balance was presented in this paper. It determines the optimum design achieving as possible as the earthwork balance condition, less
earthwork amount. In addition, it takes into consideration the design criteria for grades and vertical curve including the rate of curvature as well as the minimum grades. New mathematical relationships were derived to help the optimization process. The optimization process is searching for only one independent unknown. Therefore the consumed time is less than the traditional consumed time. The proposed approach was explained and tested using a numerical example. It was found that the proposed approach leads to an optimum vertical alignment achieving earthwork amount very close to the absolute least earthwork amount in short time. The proposed approach is useful to minimize the earthwork amount as well as the transportation cost for soil from and to the road site. The proposed approach was limited in this paper by the case of one vertical curve; however, it can be expanded in the future to the case of multiple vertical curves. This generalization can be carried out by dividing the whole road into sections and applying the optimization procedure proposed in this paper for each section. The suitable way for road dividing will be examined to achieve the least earthwork quantity for the whole road and/or any other objectives. On the other hand, the earthwork quantities in this paper are calculated based on heights of cut/fill at the road centerline only and the full cross section of the road can be considered in the future work.

REFERENCES