Abstract: In this work, we solve a nonlinear dispersive Zakharov-Kuznetsov equation by optimal homotopy asymptotic method. It is initial value problem and we test the proposed method in this case. The beauty of this method lies in the auxiliary function, \( H(q) = qC_1 + q^2C_2 + \ldots \), where \( C_1, C_2, \ldots \) are used to control and adjust the region of convergence of the solution. The obtained results are compared with those obtained by HPM and ADM. The results reveal that in small domains, the optimal homotopy asymptotic method is effective and reliable for initial value problems.

Keywords: Nonlinear partial differential equation, initial value problem, optimal homotopy asymptotic method

INTRODUCTION

The discovery of solitary waves by John Scott Russell [1] in 1834, attracted many scientists to work on this concept. Zabuski and Kruskal [2] named these solitary waves as solitons. Due to its significance in the scientific fields such as fluid dynamics, astrophysics and plasma physics substantial work has been done in this area. The KdV equation governs the height of the surface of shallow water in the presence of solitary waves. It was discovered that solitons are solutions to the Korteweg-de Vries (KdV) equation [3]. In simplest form, the KdV equation is

\[ u_t + uu_x + u_{xxx} = 0 \]

This equation appears in many areas like plasma physics, acoustic waves, heat pulses etc.

Rosenau and Hyman [4] introduced a family of full non-linear partial differential equations

\[ K(m,n):u_t + a(u^n)_x + (u^n)_{xxx} = 0, m > 0, 1 < n \leq 3 \]

to study the role of non-linear dispersion in the formation of patterns in liquid drops. They discovered a class of solitary waves with compact support and finite wavelength and they named these waves as compactons.

The best known two-dimensional generalizations of the KdV equations are the Kadomtsev-Petviashvilli (KP) equation and the Zakharov-Kuznetsov (ZK) equation. The KP equation is given by

\[ u_t + uu_x + u_{xxx} + u_{yy} = 0 \]

which characterizes small-amplitude, weakly dispersive waves on a fluid sheet and it accounts for slowly varying transverse perturbations of unidirectional KdV solitons moving along the x-direction [5]. The ZK equation in (2+1) dimensions, is given by

\[ u_t + uu_x + (u_x^2 + u_y^2)_x = 0 \]

The ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field.

Recently Ma Hong-Cai et al. [6], solved ZK equation by using the generalized auxiliary equation method and new solitary pattern, solitary wave and singular solitary wave solutions are found. B. Ganjavi et al. [7], used HPM and for solving ZK equation.

Recently, a new homotopy based method, the optimal homotopy asymptotic method (OHAM), has been introduced by Vasile Marinca et al. [8-11]. These authors used a more flexible function, named as
auxiliary function, to control the convergence of the developing series. Unlike HAM, the authors reported straightforward methods for the determination of auxiliary constants that control the convergence of the solution. They explicitly defined all the recursive relations that can be handled easily. By the virtue of the more flexible auxiliary function, this method gives more accurate and reliable results than HAM and HPM.

Javed Ali et al. [12-16] used OHAM for the solution of many problems. In [17], Daftardar-Jafari polynomials are used to enhance the accuracy of standard OHAM.

We apply OHAM to the ZK equation. The main goal of this work is to solve the ZK equation of the form ZK(m, n, k):

\[ t + \frac{a(u^n)_{xx} + b(u^*)_{xxx} + c(u^*)_{yys}}{x, x, y} = 0, \quad (m, n, k) \neq 0 \]

where a, b, c are arbitrary constants and m, n, k are integers.

**ANALYSIS OF THE METHOD**

We consider a general PDE:

\[ L(u) + g(r) + F(u) = 0, \quad r \in \Omega \]

\[ B\left( u, \frac{\partial u}{\partial n} \right) = 0, \quad n \in \Gamma \]  

(1)

where \( L \) is a linear operator, \( u \) is the unknown function of \( r \) which is to be determined, \( r \) is an n-tuple, \( g(r) \) is a source function, \( F \) is a nonlinear and \( B \) is a boundary operator.

According to OHAM we construct a homotopy:

\[ H(v(r,p) : \Omega \times [0, 1] \rightarrow \mathbb{R} \] which satisfies

\[ (1-p)[L(v(r,p)) + g(r)] = h(p)[L(v(r,p)) + g(r) + F(u(r,p))] \]

\[ B\left( v, \frac{\partial v}{\partial n} \right) = 0 \]  

(2)

where \( p \in \Omega \) and \( p \in [0, 1] \) is an embedding parameter, \( h(p) \) is a nonzero auxiliary function for \( p=0 \), \( h(0) = 0 \) and \( v(r,p) \) is an unknown function. Obviously, when \( p = 0 \) and \( p = 1 \) it holds that \( v(r,0) = u_0(r) \) and \( v(r,1) = u(r) \) respectively.

Thus, as \( p \) varies from 0 to 1, the solution \( v(r,p) \) approaches from \( u_0(r) \) to \( u(r) \), where \( u_0(r) \) is obtained from Eq (2) for \( p = 0 \) and we have

\[ L(u_0(r)) + g(r) = 0, \quad B(u_0, \partial u_0) = 0 \]  

(3)

Next we choose auxiliary function \( h(p) \) in the form

\[ h(p) = pC_1 + p^2C_2 + ... \]  

(4)

where \( C_1, C_2, ... \) are parameters to be determined. \( h(p) \) can be expressed in many forms as reported by V. Marinca et al. [8-11].

To get an approximate solution, we expand \( v(r,p,C_i) \) in Taylor’s series about \( p \) in the following manner:

\[ v(r,p,C_i) = u_0(r) + \sum_{i=1}^{m} u_i(r, C_i, C_{i+1}, ..., C_m) p^n \]  

(5)

Substituting Eq. (5) into Eq. (2) and equating the coefficient of like powers of \( p \), we obtain the following linear equations.

Zeroth order problem is given by Eq. (3) and the first order problem is given by Eq. (6):

\[ L(u_1(r)) + g(r) = C_1 F_0(u_0(r)), \quad B(u_1, \partial u_1) = 0 \]  

(6)

The mth-order equation is given by

\[ L(u_k(r)) - L(u_{k-1}(r)) = C_k F_k(u_0(r)) + \sum_{i=1}^{m} \left[ L(u_0(r)) + F_0(u_0(r)) \right] \]

\[ k = 2, 3, ..., m \]

\[ B(u_k, \partial u_k) = 0 \]  

(7)

The mth-order approximate solution is

\[ \hat{u}(r, C_i) = u_0(r) + \sum_{i=1}^{m} u_i(r, C_i, C_{i+1}, ..., C_m) \]  

(8)

Residual in this case is

\[ R(r, C_i) = L(\hat{u}(r, C_i)) + g(r) + F(\hat{u}(r, C_i)) \]  

(9)

We will have the exact solution, \( u \) if \( R = 0 \). Otherwise the residual is minimized over the domain of the problem using the methods: Least squares, Ritz or Galerkin’s. This minimization gives us the optimal values of \( C_i, i = 1, 2, 3, ... \)

We apply the Galerkin’s method for the above purpose for a fixed \( t \) and \( \hat{u} = \hat{u}(x, y) \). In accordance with this method, we solve the following system:

\[ \int_R \frac{\partial \hat{u}}{\partial C_i} = 0 \]

(10)

The values \( C_i, i = 1, 2, 3, ... \) can also be obtained viva collocation by taking fixed points \( P_i \) in the domain of the problem using (5.6) and then solving (\( R = 0, C_i \)).
APPLICATION OF OHAM

We consider the following ZK (3, 3, 3) equation with an initial value:

\[ u_t + (u^3)_x + 2(u^3)_xxx + 2(u^3)_{yxx} = 0 \]

\[ u(x, y, 0) = \frac{3}{2} \sinh \left( \frac{x + y}{6} \right) \]  \hspace{1cm} (10)

where the arbitrary constant ? ? is assumed to be 1.

By means of OHAM, taking \( H(p) = pC_1 + p^2C_2 \), we obtain the following, zeroth, first and 2\(^{nd} \) order problems:

**Zeroth-order:**

\[ \frac{\partial u_0}{\partial t} = 0, \text{ with initial condition } u_0(x, y, 0) = \frac{3}{2} \sinh \left( \frac{x + y}{6} \right) \]

**First-order:**

\[ \frac{\partial u_1}{\partial t} = (1 + C_1) \frac{\partial u_0}{\partial t} + C_2 \frac{\partial^2 u_0}{\partial y^2} + 12C_1 \left( \frac{\partial u_0}{\partial x} \right)^2 \frac{\partial u_0}{\partial y} + 12C_1 \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial u_0}{\partial x} + 12C_1 \frac{\partial^2 u_0}{\partial x \partial y} + 6C_1 \frac{\partial^2 u_0}{\partial x^2} + 36C_2 \frac{\partial u_0}{\partial x} \frac{\partial^2 u_0}{\partial x \partial y} + 6C_1 \frac{\partial^2 u_0}{\partial x \partial y} + u(x, y, 0) = 0 \]

**Second-order:**

\[ \frac{\partial u_2}{\partial t} = C_1 \frac{\partial u_0}{\partial t} + C_2 \frac{\partial u_0}{\partial y} + C_2 \frac{\partial u_0}{\partial x} + 6C_1 \frac{\partial u_0}{\partial x} + 12C_1 \frac{\partial^2 u_0}{\partial x \partial y} + 24C_1 \frac{\partial^2 u_0}{\partial y^2} + 12C_1 \frac{\partial^2 u_0}{\partial x^2} + 12C_1 \frac{\partial^2 u_0}{\partial x \partial y} + 6C_2 \frac{\partial u_0}{\partial x} + 36C_2 \frac{\partial u_0}{\partial x} \frac{\partial^2 u_0}{\partial x \partial y} + 6C_2 \frac{\partial^2 u_0}{\partial x \partial y} + 12C_1 \frac{\partial^2 u_0}{\partial x \partial y} + 6C_2 \frac{\partial^2 u_0}{\partial x \partial y} + 36C_2 \frac{\partial u_0}{\partial x} \frac{\partial^2 u_0}{\partial x \partial y} + 36C_2 \frac{\partial^2 u_0}{\partial x \partial y} + 12C_2 \frac{\partial^2 u_0}{\partial x^2} + 6C_2 \frac{\partial^2 u_0}{\partial x \partial y} + 6C_2 \frac{\partial^2 u_0}{\partial x \partial y} + 12C_2 \frac{\partial^2 u_0}{\partial x \partial y} + 36C_2 \frac{\partial^2 u_0}{\partial x \partial y} \]

The obtained solutions are:

- **Solution of zeroth-order is**
  \[ u_0(x, y, t) = \frac{3}{2} \sinh \left( \frac{x + y}{6} \right) \]

- **Solution of first-order is**
  \[ u_1(x, y, t) = \frac{3C_1}{32} \sinh \left( \frac{x + y}{2} \right) - 5 \cosh \left( \frac{x + y}{6} \right) \]

- **Solution of second-order is**
  \[ u_2(x, y, t) = \frac{15}{32} \cosh \left( \frac{x + y}{6} \right) - \frac{15}{32} C_1 \cosh \left( \frac{x + y}{2} \right) - \frac{15}{32} C_2 \cosh \left( \frac{x + y}{6} \right) \]

We consider the following second order approximation:

\[ \tilde{u}(x, y, t) = u_0(x, y, t) + u_1(x, y, t) + u_2(x, y, t) \]
Residual of this solution is 

\[ R = \bar{u} + (\bar{u})_x + 2(\bar{u})_y + 2(\bar{u})_y \]

For \( t = 0.001 \), we solve

\[
\int_0^{0.5} \int_0^{0.5} \frac{\partial \bar{u}}{\partial C_1} \, dx \, dy = 0, \quad \int_0^{0.5} \int_0^{0.5} \frac{\partial \bar{u}}{\partial C_2} \, dx \, dy = 0
\]

and obtain

\[ C_1 = -1.0279768796 \quad \text{and} \quad C_2 = 2.0002481934 \]

Using these values, the second order approximation (11) becomes

\[
\bar{u}(x,y) = 1.5000002061 \sinh \left( \frac{x + y}{6} \right) - \frac{1.8287 \times 10^{-6}}{6} \sinh \left( \frac{x + y}{6} \right) - 0.00046875496 \cosh \left( \frac{x + y}{6} \right) + 0.0008437598 \cosh \left( \frac{x + y}{2} \right) + 2.2528 \times 10^{-6} \sinh \left( \frac{x + y}{6/5} \right)
\]

For \( y = 0.1 \), we display the numerical results for solution (13) in Table 1.

Table 1: In this table, we compare OHAM solution (13) with results obtained by applying ADM and HPM. The residual for OHAM solution is computed at different mesh points indicating sufficiently good accuracy although only two iterations are performed.

Using again Eq.(12), with the interval of integration from 0 to 1, we obtain \( C_1 = 1.005832689 \) and \( C_2 = -2.023377226786 \), for \( t = 0.001 \). For these values, the second order approximation (11) becomes

\[
\bar{u}(x,y) = 2.2674 \times 10^{-6} \sinh \left( \frac{x + y}{6/5} \right) - \frac{1.8406 \times 10^{-6}}{2} \sinh \left( \frac{x + y}{2} \right) + 0.0008437598 \cosh \left( \frac{x + y}{6} \right) + 0.00046875496 \cosh \left( \frac{x + y}{6} \right) + 1.50000207 \sinh \left( \frac{x + y}{6} \right)
\]

For \( y = 0.1 \), we display the numerical results for the solution (14) in Table 2.

Using again Eq. (12), we obtain \( C_1 = -1.0279768796 \) and \( C_2 = 2.0002481934 \), for \( t = 0.01 \). For these values, Eq. (11) becomes
Table 2:
\[
\begin{array}{ccc}
 x & \text{OHAM Sol.} & \text{Residual} \\
 0.0 & 0.025377243 & -8.4394\times10^{-6} \\
 0.1 & 0.050388419 & -6.4705\times10^{-6} \\
 0.2 & 0.075415479 & -4.7138\times10^{-6} \\
 0.3 & 0.100465388 & -3.1555\times10^{-6} \\
 0.4 & 0.125545122 & -1.7864\times10^{-6} \\
 0.5 & 0.150661670 & -6.0192\times10^{-7} \\
 0.6 & 0.175822037 & 3.9804\times10^{-7} \\
 0.7 & 0.201033244 & 1.2086\times10^{-6} \\
 0.8 & 0.226302331 & 1.8196\times10^{-6} \\
 0.9 & 0.251636360 & 2.2154\times10^{-6} \\
 1.0 & 0.277042417 & 2.3742\times10^{-6} \\
\end{array}
\]

Table 3:
\[
\begin{array}{ccc}
 x & \text{OHAM Sol.} & \text{Residual} \\
 0.00 & 0.028775435 & -1.3400\times10^{-4} \\
 0.05 & 0.041295863 & -9.3262\times10^{-5} \\
 0.10 & 0.053823905 & -5.7657\times10^{-5} \\
 0.15 & 0.066360457 & -2.7151\times10^{-5} \\
 0.20 & 0.078906416 & -1.7412\times10^{-5} \\
 0.25 & 0.091462684 & 1.8542\times10^{-5} \\
 0.30 & 0.104030166 & 3.3638\times10^{-5} \\
 0.35 & 0.116609772 & 4.3456\times10^{-5} \\
 0.40 & 0.129202415 & 4.7871\times10^{-5} \\
 0.45 & 0.141809013 & 4.6727\times10^{-5} \\
 0.50 & 0.154430487 & 3.9832\times10^{-5} \\
\end{array}
\]

Table 4:
\[
\begin{array}{ccc}
 x & \text{OHAM Sol.} & \text{Residual} \\
 0.00 & 0.066360457 & -2.7151\times10^{-5} \\
 0.05 & 0.078906416 & -1.7412\times10^{-6} \\
 0.10 & 0.091462684 & 1.8542\times10^{-5} \\
 0.15 & 0.104030166 & 3.3638\times10^{-5} \\
 0.20 & 0.116609772 & 4.3456\times10^{-5} \\
 0.25 & 0.129202415 & 4.7871\times10^{-5} \\
 0.30 & 0.141809013 & 4.6727\times10^{-5} \\
 0.35 & 0.154430487 & 3.9832\times10^{-5} \\
 0.40 & 0.167067765 & 2.6959\times10^{-5} \\
 0.45 & 0.179721776 & 7.8447\times10^{-6} \\
 0.50 & 0.192393456 & -1.7812\times10^{-5} \\
\end{array}
\]

Table 5:
\[
\begin{array}{ccc}
 x & \text{OHAM Sol.} & \text{Residual} \\
 0.00 & 0.129202415 & 4.7871\times10^{-5} \\
 0.05 & 0.141809013 & 4.6727\times10^{-5} \\
 0.10 & 0.154430487 & 3.9832\times10^{-5} \\
 0.15 & 0.167067765 & 2.6959\times10^{-5} \\
 0.20 & 0.179721776 & 7.8447\times10^{-6} \\
 0.25 & 0.192393456 & -1.7812\times10^{-5} \\
 0.30 & 0.205083745 & -5.0354\times10^{-5} \\
 0.35 & 0.217793589 & -9.0163\times10^{-6} \\
 0.40 & 0.230523937 & -1.3767\times10^{-4} \\
 0.45 & 0.243275745 & -1.9334\times10^{-4} \\
 0.50 & 0.256049972 & -2.5771\times10^{-4} \\
\end{array}
\]

Fig. 2: For solution (15) at \( y = 0.1 \)

\[
\tilde{u}(x, y) = 0.0084462 \cosh\left(\frac{x + y}{2}\right) + 1.50002 \sinh\left(\frac{x + y}{6}\right) - 0.00469233 \cosh\left(\frac{x + y}{6}\right) - 0.000192256 \sinh\left(\frac{x + y}{2}\right) + 0.000236837 \sinh\left(\frac{x + y}{6/5}\right)
\]

For \( y = 0.1 \), we display the numerical results for the solution (15) in Table 3.

For \( y = 0.25 \), we display the numerical results for the solution (15) in Table 4.

For \( y = 0.5 \), we display the numerical results for the solution (15) in Table 5.

For \( y = 1 \), we display the numerical results for the solution (15) in Table 6.

Fig. 3: Plot of the residual (for solution 15) at \( t = 0.01 \) and \( y = 0.1 \)
Table 6:

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<th>OHAM Sol.</th>
<th>Residual</th>
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CONCLUSION

In this work, we have solved an initial value problem i.e., the ZK-equation with an initial condition to examine the efficiency of the method. We found that this recently developed method has the potential to handle a wide class of initial value problems. In this method, all the recursive relations have been clearly and explicitly defined and for the determination of auxiliary parameters which control the convergence, a straight forward approach has been used.

REFERENCES