On the Homology Theory of Operator Algebras

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Abstract: The homology (cyclic and dihedral) of unital and involuted Banach algebra are studied. The extension of results in [6] for dihedral case is studied. Also the Dihedral homolog of commutative involutive unital Banach algebra, Laurent polynomial algebra and involutive tensor Banach algebra are studied. The main results are theorems (2.2), (3.1) (3.2), (4.1) and (4.2).

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INTRODUCTION

Suppose that A is a Banach algebra and [X] be a class of polynomials defined by the norm:

\[ \|x\| = \sup \{X(x_1, x_2, ..., x_n) : \|x_i\| \leq 1 \} \]  

(1.1)

Then \( A[x] \) is Banach polynomial algebra with the multiplication of coefficient of polynomials. Firstly we recall some definitions and facts about dihedral homology and its properties [1, 3]. Let A be unital Banach algebra with an involution over field K and \( C_*(A), C_{**}(A) \) bare Hochschild complex and bicomplex, respectively. The homology group of these complexes give the Simplicial (Hochschild) and cyclic homology \( H_*(A), HC_{**}(A) \) of algebra A. Consider the action of the group \( \mathbb{Z}/2 \) on Connes-Tsygan bicomplex \( C_{**}(A) \) by means of operators

\[ e_r : a_0 \otimes a_1 \otimes ... \otimes a_n \rightarrow (-1)^{\frac{n(n-1)}{2}} e.a_0 \otimes a_n \otimes ... \otimes a_1, e = \pm 1 \]  

(1.2)

The dihedral homology \( HD_*(A) \) of algebra A is defined by the hyper homology \( H_*(\mathbb{Z}/2, C_{**}(A)) \) of group \( \mathbb{Z}/2 \) with a coefficient in bicomplex \( C_{**}(A) \) [5].

Consider chain maps

\[ : C(A) \otimes C(B) \rightarrow C_*(A \otimes B) \]  

(1.3)

\[ \perp : C_*(A \otimes B) \rightarrow C_*(A) \otimes C_*(B) \]  

(1.4)

where \( C_*(A), C_*(B) \) are Hochschild complexes of algebras A and B, respectively, such that

\[ e_r(a_0 \otimes a_1 \otimes ... \otimes a_n \rightarrow (-1)^{\frac{n(n-1)}{2}} e.a_0 \otimes a_n \otimes ... \otimes a_1, e = \pm 1 \]  

(1.5)

\[ \perp((a \otimes b) \otimes (a_n \otimes b_n) \otimes ... \otimes (a \otimes b_n)) = \sum_{\sigma} (a_n, ..., a_1) \otimes (b_n, ... , b_1) \]  

(1.6)

It's known that

- The chain maps \( \perp \) and \( \perp \) induce isomorphism of Homology and the inverse is true [2]
- Clearly if A and B involution k-algebra, we can define an the involution of tensor algebra \( A \otimes B \) as follows: \( (a \otimes b)^* = a^* \otimes b^* \), \( a \in A, b \in B \)
The action of group \( \mathbb{Z}/2 \) on the Hochschild complex \( C_\bullet(A) \) of algebra \( A \) by means of (1.2) induces action of \( \mathbb{Z}/2 \) on the Hochschild homology \( \text{HH}_\bullet(A) \) of algebra \( A \).

The spectral sequence of bicomplex \( C_{\bullet\bullet}(A) \) is given by

\[
j_{2i}^j \equiv \begin{cases} 1 & i \equiv 0 \text{(mod 2)} \\ 0 & i \equiv 1 \text{(mod 2)} \end{cases}
\]

with the differentials

\[
d_{ij}^k = \begin{cases} B, & i \equiv 0 \text{(mod 2)}, i > 0 \\ 0, & i \equiv 1 \text{(mod 2)}, i = 0 \end{cases}
\]

where \( B \) is Connes’s operator [5].

The action of group \( \mathbb{Z}/2 \) on the spectral sequence \( E_{2i}^j(A) \) generates to the action of \( \mathbb{Z}/2 \) on every term of \( \text{HH}_\bullet(A) \), \( 2 < S \leq \infty \).

**POLYNOMIAL ALGEBRA AND ITS DIHEDRAL HOMOLOGY GROUP**

In this part we discuss the action of \( \mathbb{Z}/2 \) of Hochschild homology by considering the chain operators - and \( \bot \) by the following lemma

**Lemma (2.1):** Let \( \alpha \in \text{H}_n(A) \) and \( \beta \in \text{H}_m(B) \). Then

\[
(\epsilon \delta) \tau((\alpha \otimes \beta)(a_0 \otimes 1) \otimes \ldots (a_{n-1} \otimes 1) \otimes (b_0 \otimes 1) \otimes \ldots (b_{m-1} \otimes 1)) =
\]

\[
\frac{1}{m+n} \tau((\epsilon \delta)(a^* \otimes a_{n-1}^* \otimes \ldots \otimes a_0^*) \otimes (b^* \otimes b_{m-1}^* \otimes \ldots \otimes b_0^*)) =
\]

Thus \( \tau(\alpha) \otimes (\epsilon \delta) \tau(\beta) = \downarrow \tau(\alpha \otimes \beta) \) and hence \( \tau(\alpha) \tau((\epsilon \delta)(\beta)) = (\epsilon \delta) \tau(\alpha \otimes \beta) \).

The main theorem of this part is the following assertion

**Theorem (2.2):** Let \( A \) be commutative involutive unital Banach algebra over \( K \). Then

\[
\epsilon \text{HD}_n(A[x]) = \epsilon \text{HD}_n(A) \oplus \epsilon \text{HR}_n^\infty(A), \quad e = \pm 1
\]

where

\[
\epsilon \text{HR}_n^\infty(A) = \epsilon \text{H}_n(\mathbb{Z}/2;C_\bullet(A)) \oplus \epsilon \text{H}_n(\mathbb{Z}/2;C_\bullet(A)) \oplus \ldots
\]
Theorem (2.3): If the algebra A has involution *, then the algebra $A[x]$ also has involution i.e.

$$
\left( \sum_{i} a_i x_i \right)^* = \sum_{i} a^*_i x_i, \quad a_i \in A \quad (2.2)
$$

Notice that

- $A[x] = A \otimes k[x]:ax' \leftrightarrow a \otimes x'$, is involutive isomorphism (on k an involution is trivial)
- Hochshid Homology $H_*(k[x])$ isomorphic to exterior algebraic form in $K[x]$ and Connes’s operator B in Hochschild Homology corresponds to exterior differential form $d$ [2].

Since $A[x] = A \otimes K[x]$, then

$$
H_*(A[x]) = H_*(A) \otimes H_*(k[x])
$$

Under this isomorphism the Connes's operator $B$ acts on $H_*(A) \otimes H_*(k[x])$ by means

$$
B(\alpha \otimes \beta) = B(\alpha) \otimes \beta + (-1)^{deg} \alpha \otimes B(\beta)
$$

[2]. Since

$$
H(k[x]) = \begin{cases}
\Omega^0, & i = 0 \\
\Omega^1, & i = 1 \\
0, & i > 1
\end{cases} \quad (2.3)
$$

where $\Omega^i$-exterior algebraic form in $K[x]$, then the second term in spectral sequence $\{E^{\ast}_n, A[x]\}$ takes the form

$$
E^{2}_0(A[x]) = \begin{cases}
H_j(A) \otimes \Omega^0 \otimes H_{j-1}(A) \otimes \Omega^1, & i = 0 \pmod{2} \\
0, & i = 1 \pmod{2}
\end{cases} \quad (2.4)
$$

$$
d^{2}_{j}(\alpha \otimes \beta) = \begin{cases}
B(\alpha) \otimes \beta + (-1)^{deg} \alpha \otimes d(\beta), & i = 0 \pmod{2}, i > 0, \\
0, & i \neq 0 \pmod{2},
\end{cases} \quad (2.5)
$$

In otherwise, the cohomology of complex

$$
H(A[x]) \xrightarrow{d^1} \ldots \xrightarrow{d^1} H_1(A[x]) \xrightarrow{d^1} \ldots
$$

isomorphic to the cohomology of complex

$$
H_*(A[x]) \xrightarrow{b} H_*(A[x]) \xrightarrow{b} \ldots \xrightarrow{b} H_1(A[x]) \xrightarrow{b} \ldots
$$

Using Kunth formula [2], we get:

$$
H^*(H,(A) \otimes \Omega^0) = H^*(H,(A) \otimes H(\Omega^0) = H^*(H,(A))
$$

since

$$
H^*(\Omega^0) = \begin{cases}
k, & \ast = 0 \\
0, & \ast > 0
\end{cases}
$$

The term $E^{1}_0(A[x])$ has the form

$$
E^{1}_0(A[x]) = \begin{cases}
\ker B / \text{Im} B, & i = 0 \pmod{2}, i > 0 \quad (2.6) \\
0, & i = 1 \pmod{2}
\end{cases}
$$

Note, that the groups $E^{1}_0(A[x]), i > 0$ isomorphic to groups $E^{1}_0(A), i > 0$

Clearly the groups $E^{1}_0(A[x]), i > 0$ isomorphic to group

$$
(H_j(A) / \text{Im}(H_j(A)) \otimes H_j(A) \otimes \Omega^0 / k = H^*(H,(A))
$$

where $\Omega^0/k$ is space of algebraic 0-form factored by k. Hence

$$
E^{1}_0(A[x]) = \begin{cases}
E^{1}_0(A), & i = 0 \pmod{2}, i > 0 \quad (2.7) \\
0, & i = 1 \pmod{2}
\end{cases}
$$

The spectral sequence $E^{1}_0(A[x])$ without term

$$
H^*_j(A) = H_j(A) \otimes \Omega^0 / k
$$
tends to the cyclic homology group $HC_*(A)$ and hence

$$
HC_*(A[x]) = HC_*(A) \otimes H_*(A)
$$

Notice that the group $\mathbb{Z}/2$ acts on groups $E^{1}_0(A[x]), i > 0$, by means of operators $r$. The group $\mathbb{Z}/2$ also acts on groups:

$$
E^{3}_0(A[x]) = H_j(A) / \text{Im}(H_j(A)) \otimes H_j(A) \otimes \Omega^0 / k \quad (2.8)
$$

by means of

$$
\varepsilon r(\alpha \otimes \beta \otimes w) = \varepsilon r(\alpha) \otimes \varepsilon r(\beta) \otimes \varepsilon r(w) \quad (2.9)
$$

Where
\[ \alpha \in H_j(A)/B(H_{j-1}(A)), \beta \in H_j(A), \omega \in \Omega^0/k \]

Since \( A \) is Banach algebra, then the spectral sequence \( \{ \xi E_n(A[x]) \} \) expressed by

\[ \xi E^2_{pq}(A[x]) = \xi H_p(\mathbb{Z}/2, HC_q(A)) \quad (2.10) \]

By factoring the group \( HC(A) \otimes H^\omega(A) \) by the operator \( (1 - \epsilon) \), we get the statement of theorem

Let \( A \) be commutative, involution, unital \( K \)-Banach algebra, then

\[ e HD_{\ast}(A) = e HD_{\ast}(A) \otimes e HD_{\ast-1}(A) \otimes HR_{\ast}^\omega(A) \]

**Proof:**

\[ \left( \sum_{i} a_i x^i + \sum_{j} b_j x^{-j} \right)^* = \sum_{i} a_i^* x_i + \sum_{j} b_j^* x^{-j}, \quad a_i, b_j \in A \]

Also there is an involution isomorphism \( K \)-algebra

\[ e HD_n(A[x,x^{-1}]) = e HD_n(A) \otimes e HD_{n-1}(A) \otimes HR_n^\omega(A) \]

The Hochshild homology \( H_\ast(k[x,x^{-1}]) \) isomorphic to exterior algebraic form in \( k[x,x^{-1}] \). The differential \( d \) of exterior form \( \Omega^1 \), \( i \geq 0 \) on \( k[x,x^{-1}] \) corresponds to connes's operator \( B \) on Hochshild Homology \( H_\ast(k[x,x^{-1}]) \). Since

\[ A[x,x^{-1}] = \otimes_k k[x,x^{-1}] \]

then the Connes's operators \( B \) acts on \( H_\ast(A) \otimes H_\ast(k[x,x^{-1}]) \) by means of

\[ B(\alpha \otimes \beta) = (1 \otimes \beta + (-1)^{deg \alpha} \alpha \otimes B(\beta)) \]

Since

\[ H_i(k[x,x^{-1}]) = \begin{cases} \Omega^0, & i = 0 \\ \Omega^1, & i = 1 \\ 0, & i > 1 \end{cases} \]

then the first term of spectral sequence \( \{ \xi E_n(A[x,x^{-1}]) \} \) has form

\[ \xi E^3_{ij}(A[x,x^{-1}]) = \begin{cases} H_j(A) \otimes \Omega^0 \otimes H_{j-1}(A) \otimes \Omega^1 \\ 0, & i \equiv 0(\text{mod} 2) \\ 0, & i \equiv 1(\text{mod} 2) \end{cases} \]

Calculating the cohomology of the complex

\[ A[x,x^{-1}] \xrightarrow{d^2} \cdots \xrightarrow{d^2} H_0(A[x,x^{-1}]) \xrightarrow{d^2} H^0(H_\ast(A) \Omega^\ast) \]

then the groups \( \{ \xi E_n(A[x,x^{-1}]) \} \) has the form

\[ \xi E^3_{ij}(A[x,x^{-1}]) = \begin{cases} H_j(A) \otimes \Omega^0 \otimes H_{j-1}(A) \otimes \cdots \otimes \Omega^1 \otimes \Omega^1, & i = 0 \\ 0, & i \equiv 0(\text{mod} 2), i > 0 \end{cases} \]

Clearly the group \( e E_\ast(A[x,x^{-1}]) \) isomorphic to groups
In Spectral sequence \( \{ E^q_{ij}(A[x,x^-]) \} \), the group \( E^1_{ij}(A[x,x^-]) \) without the term
\[
\mathcal{H}^i(A) = H(A) \oplus \Omega^i/k
\]
converges to group \( HC_n(A) \oplus HC_{n-i}(A) \oplus HH_i^*(A) \)

Further, on groups \( E^i_{ij}(A[x,x^-]), i > 0 \), the group \( \mathbb{Z}/2 \) acts by means of operators \( ^e r \otimes^e r_i \) and on \( E^1_{ij}(A[x,x^-]) \) as follows
\[
e^e r(\alpha \otimes (\beta \otimes w) \otimes \gamma) = ne^e (\varepsilon(\alpha) \otimes^e r(\beta) \otimes^e r(w) \otimes^e r(\gamma)
\]
where
\[
\alpha \in H_j(A)/B(H_{j-1}(A)), \quad \beta \in H_j(A), \quad w \in \Omega^0/k, \quad \gamma \in H_{j-1}(A)/B(H_{j-2}(A))
\]

For Banach algebra the spectral sequence \( \{ E^q_{ij}(A[x,x^-]) \} \) degenerated in terms

For \( l = (i, \ldots, i) \) put \( l^p = i + \ldots + i_s \nabla^p = x^k \oplus_B I_B \overset{10h_{2m}}{\longrightarrow} x^k \oplus_B \sum_{i=0} \left( x^{m(n-1)} \oplus I + (x \otimes x^i \otimes \ldots x^i \otimes x^i) \right) \)
\[
\sum_{i=0} \left( x^{m(n-1)} \oplus I + (x \otimes x^i \otimes \ldots x^i \otimes x^i) \right) \overset{2m(2m+1)}{\rightarrow} \sum_{i=0} \left( x^{m(n-1)} \oplus I + (x \otimes x^i \otimes \ldots x^i \otimes x^i) \right)
\]

Thus on the Hochchinld homology \( H_{2m}(A) \) the operator \( ^e r_{2m} \) acts with multiplication on \((-1)^{n}e^e r_i \).

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\[
\sum_{i=0} \left( x^{m(n-1)} \oplus I + (x \otimes x^i \otimes \ldots x^i \otimes x^i) \right) \overset{2m(2m+1)}{\rightarrow} \sum_{i=0} \left( x^{m(n-1)} \oplus I + (x \otimes x^i \otimes \ldots x^i \otimes x^i) \right)
\]

Thus on the Hochchinld homology \( H_{2m}(A) \) the operator \( ^e r_{2m} \) acts with multiplication on \((-1)^{n}e^e r_i \).

Using the spectral sequence of bicomplex \( \mathcal{C}_*^*(A) \) we have
\[
E^2_{ij}(A) = \begin{cases} 
  k^n, & i = 0 \text{(mod} 2), \quad j = 0 \\
  k^{n-1}, & i = 2 \text{(mod} 2), \quad i > 0 \\
  0, & i = 1 \text{(mod} 2). 
\end{cases}
\]

In [6] Massuda has shown that \( B : H_{2m-1}(A) \to H_{2m}(A) \) its zero homomorphism and \( B : H_{2m}(A) \to H_{2m-1}(A) \) is an isomorphism but \( B : H_j(A) \to H_k(A) \) is epimorphism.

In \( E^2_{ij}(A) \) the spectral sequence \( E^2_{ij}(A) \) is given by
Since \( A \) is Banach \( k \)-algebra (\( \text{char}(k)=0 \)), then there is a spectral sequence of second term. Now by factoring the group

\[
H_{\bullet}(A) = \bigoplus_{p+q} E_{ij}^2(A)
\]

by \( \text{Im}(1-\epsilon) \) we get

\[
E_{ij}^3(A)/(1-1)_{ij} = \begin{cases} 
    k^n, & i = j = 0 \\
    k^{n-1}, & 0, j = 0, i \equiv 0 \pmod{2} \\
    0, & j = 0, i \equiv 0 \pmod{2}
\end{cases}
\]

Thus we have

\[
1^H_D(k[x]/(x^n)) = \begin{cases} 
    k^i, & i = 0 \pmod{4} \\
    k, & i = 2 \pmod{4} \\
    0, & i = 1 \pmod{4}
\end{cases}
\]

\[
-1^H_D(k[x]/(x^n)) = \begin{cases} 
    k^{n-1}, & i = 2 \pmod{4} \\
    0, & i \not\equiv 2 \pmod{4}
\end{cases}
\]

**DIHEDRAL HOMOLOGY OF TENSOR ALGEBRA**

Let \( A \) involution \( k \)-Banach algebra, where \( k \) is a real or complex number. Consider tensor algebra \( T_A(A) \) with involutive \( A \)-bimodule. Recall that \( A \)-bimodule has involutive, if there is an automorphism \( *:M \rightarrow M \) of order two such that \( (amb)^* = b^*m^*a^* \), \( a,b \in A \). Suppose that \( P_\bullet \)-chain complex of \( A \)-bimodules with an involutive. Consider the complex

\[
S^k(A,P_\bullet) = A \otimes_{A \otimes A^*} P_\bullet \otimes (k+1)
\]

Acting on \( S^k(A,P_\bullet) \) by two automorphisms

\[
t_k(P_0 \otimes \ldots \otimes P_k) = (-1)^k P_\bullet \otimes P_0 \otimes \ldots \otimes P_{k-1} \tag{4.1}
\]

\[
\epsilon \theta_k(P_0 \otimes \ldots \otimes P_k) = (-1)^k \epsilon P_\bullet \otimes \ldots \otimes P_0 \otimes P_k \tag{4.2}
\]

Where

\[s = (\deg P_0) \left( \sum_{i=0}^{k-1} \deg P_i \right)\]

\[\epsilon = \left( k + \sum_{i=0}^{k} \deg P_i - 1 \right) / 2\]

The automorphism \( s_t \) and \( \theta_t \) represent group dihedral \( D_{k+1} \) of order \( 2(k+1) \). If \( P_\bullet \) be free involutive \( A \)-bimodule of resolution involutive \( A \)-bimodule \( M \), then the complex \( S^k(A,P_\bullet) \) can be written as follows \( S^k(A,M) \).

**Theorem (4.1):** Let \( M \)-involutive \( A \)-bimodule and

\[
\text{Tor}_i^A(M,M) = 0, i > 0
\]

Then

\[
\epsilon^H_D((T_A(A)) = \epsilon^H_D(A) \oplus (k) \otimes (D_{k+1})^S_k(A,M).
\]

**Proof:** The long exact sequence of relative dihedral homology [4] for sequence of involutive homeomorphism algebra \( A \rightarrow T_A(M) \rightarrow 0 \), is given by

\[
\cdots \rightarrow \epsilon^H_D(A) \rightarrow T_A(M) \rightarrow 0 \rightarrow \epsilon^H_D(T_A(M) \rightarrow 0) \rightarrow \epsilon^H_D(T_A(M) \rightarrow 0) \rightarrow \cdots
\]

Since \( A \) is the direct sum in \( T_A(M) \), then the long exact sequence splits and hence

\[
\epsilon^H_D((T_A(M)) = \epsilon^H_D(A) \oplus \epsilon^H_D(A \rightarrow T_A(M))
\]

we show that

\[
\epsilon^H_D(A \rightarrow T_A(M)) = \bigoplus_{k=0}^\infty \epsilon^H_D(D_{k+1},S^k(A,M))
\]

Really

\[
T_A(P_\bullet) / (A + [T_A(P_\bullet),T_A(P_\bullet)] + \text{Im}(1-\epsilon))
\]

\[
= \bigoplus_{k=0} \epsilon^{k+1} / ([T_A(P_\bullet),T_A(P_\bullet)] + \text{Im}(1-\epsilon))
\]

Since

\[
\epsilon r(P_0 \otimes \ldots \otimes P_k) = (-1)^{\sum_{i=0}^k \epsilon P_0^* \otimes \ldots \otimes P_k^*}
\]

\[= (-1)^{\sum_{i=0}^k \epsilon P_k^* \otimes \ldots \otimes P_1^* \otimes P_0^*}
\]

Where
\( \psi = (k + \text{deg}(P_0 \otimes P_1 \ldots \otimes P_k))(k + \text{deg}(P_0 \otimes \ldots \otimes P_k) - 1)/2 \) then, we have an isomorphism

\[
\bigoplus_{k=0}^{\infty} (P_\bullet^{(k+1)})/([T_A(P_\bullet), T_A(P_\bullet)] + \text{Im}(1 - \epsilon)) = \bigoplus_{k=0}^{\infty} A(A \otimes A^\vee) \oplus (\text{Im}(1 - t_k) + \text{Im}(1 - \epsilon \theta_k))
\]

The homology of chain complex

\[
\bigotimes_{k=0}^{\infty} A(A \otimes A^\vee) \oplus (\text{Im}(1 - t_k) + \text{Im}(1 - \epsilon \theta_k))
\]
in exactly coincide with \( \bigoplus_{k=0}^{\infty} H_*^{(k+1)}(D_{k+1}, S^{(k)}(k,v)) \).

If A is augmented k-algebra with involution then,

\[
\epsilon \text{HD}_*(A) = \ker(\epsilon \text{HD}_*(A) \rightarrow \epsilon \text{HD}_*(k)) \quad (4.3)
\]

**Corollary (4.2):** Let V-Banach space with an involution

\[
\epsilon \text{HD}_*(T(V)) = \bigoplus_{k=0}^{\infty} H_*(D_{k+1}, V^{(k+1)})
\]

**Proof:**

\[
\epsilon \text{HD}_*(T_k(V)) = \epsilon \text{HD}_*(k) \oplus (\bigoplus_{k=0}^{\infty} H_*(D_{k+1}, S^{(k)}(k,v)))
\]

and following isomorphism

\[
\bigoplus_{k=0}^{\infty} H_*(D_{k+1}, S^{(k)}(k,v)) \cong \bigoplus_{k=0}^{\infty} H_*(D_{k+1}, V^{(k+1)}) \quad (4.4)
\]

Really \( S^{(i)}_*(k,v) = R^{(i+1)}_\bullet \), where \( R_\bullet \) is free k-module resolution of k-module V. We can get an isomorphism (4.4) by considering the sequence \( V \rightarrow 0 \rightarrow 0 \rightarrow \ldots \) from the resolution \( R_\bullet \). The corollary is proved.

**REFERENCES**