R-valued Ideals of Ordered Hemirings

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Abstract: The notion of R-valued ideals of an ordered hemiring is considered and some of its properties are given, where R is a commutative multiplicative idempotent bounded CLO-hemiring in which multiplication distributes over arbitrary joins. It is shown that the set of all R-valued ideals of a positive ordered hemiring forms an additionally idempotent bounded CLO-hemiring and that some ordered hemirings can be embedded in it.

Key words: Ordered hemiring (semiring); Poe-hemiring (Poe-semiring); CLO-hemiring(semiring).

INTRODUCTION

Given a non-empty set X, a fuzzy subset of X, by definition, is an arbitrary mapping \( f : X \rightarrow R \) where R is the unit interval \([0, 1]\) of the real line. If the set X bears some structure, one may distinguish some fuzzy subsets of X in terms of that additional structure. This important concept of a fuzzy set was first introduced by Zadeh [1] in his ground-breaking work, which enlarged the concept of a crisp subset. The theory of fuzzy sets has provided a useful mathematical tool for describing the behavior of systems that are too complex or ill-defined to admit precise mathematical analysis by classical methods and tools. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to artificial intelligence, computer science, control engineering, expert systems, management science, operations research, logic, set theory, group theory, groupoids, real analysis, measure theory, topology, and others. Many notions of mathematics are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In recent years, there has been considerable interest in the connections between fuzzy sets and algebraic structures theory. The readers are referred to [2-16].

In the original conception given in Zadeh [1], the unit interval \([0, 1]\) was equipped with the operations \( \lor \) (maximum) and \( \land \) (minimum). An important extension of this situation, however, was realized when the latter was replaced by an arbitrary triangular norm in the sense of Menger [17], namely an associative operation \( * \) on \([0, 1]\) satisfying the condition that \([(0, 1], \lor, *)\) is a semiring. Similarly, we have the notion of a triangular conorm on \([0, 1]\) satisfying the condition that \([(0, 1], \land, *)\) is a semiring. In [18], Menger interpreted triangular norms in the context of continuum physics as rules for generating new probabilistically-determined objects from existing ones in the psychophysical continuum space. In [19], Butnariu and Klement used triangular norms for interaction rules for economic agents in fuzzy games.

Goguen [20] extended Zadeh’s construction by replacing unit interval with an arbitrary bounded distributive lattice, a lead which has been followed by many others (see [21,22]). One may note that any bounded distributive lattice \( (L, \lor, \land) \) is a commutative multiplicatively idempotent bounded lattice-ordered semiring in
which multiplication distributes over arbitrary joins if we take \( x + y = x \lor y \) and \( x \cdot y = x \land y \) for all \( x, y \in R \).

This paper investigates ideal theory of an ordered hemiring in a broader framework of the \( R \)-valued function, where \( R \) is a commutative multiplicatively idempotent bounded CLO-semiring in which multiplication distributes over arbitrary joins. It shows that the set of all \( R \)-valued ideals of a positive ordered hemiring forms an additively idempotent bounded CLO-hemiring and that some ordered hemirings can be embedded in it.

**Preliminaries**

In this section, we summarize some basic concepts (see [23-26]) which will be used throughout the paper.

A *semiring* is an algebraic system \((R, +, \cdot)\) consisting of a non-empty set \( R \) together with two binary operations on \( R \) called addition and multiplication (denoted in the usual manner) satisfying the following conditions:

1. \((R, +)\) is a commutative monoid with identity element \( 0_R \);
2. \((R, \cdot)\) is a monoid with identity element \( 1_R \);
3. Multiplication distributes over addition from either side;
4. \(0_R \cdot x = x \cdot 0_R = 0_R\) for all \( x \in R \);
5. \(0_R \neq 1_R\).

If we do not have a multiplicativity identity \( 1_R \), then the structure is called a *hemiring*. \( R \) is *commutative* if \((R, \cdot)\) is commutative. A semiring (resp., hemiring) \((R, +, \cdot)\) is called *additively idempotent* if \( x + x = x \) for all \( x \in R \), and *multiplicatively idempotent* if \( x \cdot x = x \) for all \( x \in R \). For the sake of simplicity, we shall omit the symbol \( "\cdot"\), writing \( xy\) for \( x \cdot y \) (\( x, y \in R \)).

A semiring (resp., hemiring) \((R, +, \cdot)\) is called an *ordered semiring* (resp., hemiring) if and only if there exists a partial relation \( \preceq \) on \( R \) satisfying the following conditions for elements \( x, y \) and \( z \) of \( R \):

1. if \( x \leq y \) then \( x + z \leq y + z \);
2. if \( x \leq y \) and \( z \geq 0_R \) then \( xz \leq yz \) and \( zx \leq zy \).

In particular, an ordered semiring (resp., hemiring) is called a *totally ordered semiring* (resp., hemiring) if \((R, \leq)\) is a totally ordered set. And an ordered semiring (resp., hemiring) is called *positive* if \( x \geq 0_R \) for all \( x \in R \).

A semiring (resp., hemiring) \((R, +, \cdot)\) is a *complete-lattice-order semiring* (resp., hemiring), denoted as CLO-semiring (resp. CLO-hemiring), if and only if \( R \) has the structure of a complete lattice satisfying the conditions that \( x + y = x \lor y \) and \( xy \leq x \land y \) for all \( x, y \in R \).

Note that if the lattice structure of a CLO-semiring \( R \) is bounded, then the maximal element and minimal element must be \( 1_R \) and \( 0_R \), respectively.

Let \((T, \oplus, \odot, \preceq)\) and \((R, +, \cdot, \preceq)\) be ordered hemirings, \( f : T \to R \) a mapping from \( T \) into \( R \). \( f \) is called *isotone* if \( x, y \in T, x \preceq y \) implies \( f(x) \preceq f(y) \). \( f \) is said to be *inverse isotone* if \( x, y \in T, f(x) \preceq f(y) \) implies \( x \preceq y \) (each inverse isotone mapping is \( (1-1) \)). \( f \) is called a *homomorphism* if it is isotone and satisfies \( f(x + y) = f(x) + f(y) \) and \( f(x \odot y) = f(x)f(y) \) for all \( x, y \in T \). A homomorphism is called an *isomorphism* if it is surjective and inverse isotone \( T \) and \( R \) are called *isomorphic* if there exists an isomorphism between them. \( T \) is *embedded in \( R \) if \( T \) is isomorphic to a subset of \( R \), i.e., if there exists a mapping \( f : T \to R \) which is homomorphism and inverse isotone.

In the sequel, unless otherwise stated, \((T, \oplus, \odot, \preceq)\) and \((R, +, \cdot, \preceq)\) denote a positive ordered hemiring and a commutative multiplicatively idempotent bounded CLO-semiring in which multiplication distributes over arbitrary joins, respectively.

A non-empty subset \( A \) of \( T \) is called an ideal of \( T \) if it satisfies the following conditions:

1. \( A \oplus A \subseteq A \);
2. \( T \odot A \cup A \odot T \subseteq A \);
3. if \( x \in A \) and \( T \ni y \preceq x \) then \( y \in A \).

For \( A \subseteq T \), we denote

\[ \{ A \} = \{ x \in T | x \preceq y \text{ for some } y \in A \} .\]

Note that condition (3) is equivalent to the condition \( \{ A \} = A \). Let \( x \in T \). It is obvious that \( I(x) = (N x \oplus T \odot x \oplus x \odot T \oplus T \odot x \odot T) \), where \( N = \{ 0, 1, 2, \ldots \} \), is the principal ideal of \( T \) generated by \( x \). In particular, \( I(x) = (N x \oplus T \odot x) \) if \( T \) is commutative.

Denote by \( R^T \) the set of all functions from \( T \) into \( R \). For any \( A \subseteq R \), the *characteristic function* of \( A \), denoted by \( \chi_A \), is defined by

\[ \chi_A(x) = \begin{cases} 1_R & \text{if } x \in A, \\ 0_R & \text{otherwise} \end{cases} \]

for all \( x \in R \). A function \( f \in R^T \) of the form

\[ f(y) = \begin{cases} r & \text{if } y = x, \\ 0_R & \text{otherwise} \end{cases} \]

is said to be a \( R \)-valued point with support \( x \) and value \( r \) and is denoted by \( x_r \), where \( r \in R \setminus \{ 0_R \} \).

For \( f, g \in R^T \), the intersection of \( f \) and \( g \), denoted by \( f \cap g \), is defined by \( (f \cap g)(x) = f(x) \land g(x) \) for all \( x \in T \).

**\( R \)-Valued Ideals of an Ordered Hemiring**

Let \( f, g \in R^T \). We introduce the *sum, product and intrinsic product* of \( f \) and \( g \) as follows.
Definition 1. Let \( f, g \in \mathbb{R}^T \). The sum, product, and intrinsic product of \( f \) and \( g \), denoted by \( f \boxplus g \), \( f \boxtimes g \) and \( f(\sqcap)g \), are defined by: \( \forall x \in T \)

\[
(f \boxplus g)(x) = \bigvee_{y, z \in T, x \leq y \cap z} f(y)g(z)
\]

\[
(f \boxtimes g)(x) = \left\{ \begin{array}{ll}
\bigvee_{y, z \in T, \ x \leq y \cap z} f(y)g(z) & \text{if } \exists y, z \in R \\
0_{R} & \text{otherwise}
\end{array} \right.
\]

and

\[
(f(\sqcap)g)(x) = \left\{ \begin{array}{ll}
\bigvee_{x \leq \sum_{i=1}^{m} y_i \cap z_i} \prod_{i=1}^{m} f(y_i)g(z_i) & \text{if } \exists y_i, z_i \in T \\
x \leq y_1 \cap z_1 \cap \cdots \cap y_m \cap z_m, & \text{such that} \\
0_{R} & \text{otherwise}
\end{array} \right.
\]

Define an order relation \( \preceq \) on \( \mathbb{R}^T \) by: \( \forall f, g \in \mathbb{R}^T \)

\[
f \preceq g \iff f(x) \leq g(x) \text{ for all } x \in T.
\]

Lemma 2. Let \( f, g, h \in \mathbb{R}^T \). Then

(1) \( (f \boxtimes g) \boxtimes h = f \boxtimes (g \boxtimes h) \) and \( (f \boxtimes g)(\sqcap)h = f(\sqcap)(g \boxtimes h) \).

(2) \( f \boxtimes g \boxtimes h = f \boxtimes (g \boxtimes h) \).

(3) \( f \boxtimes g \preceq f(\sqcap)g \).

(4) \( 0_{\sqcup} \boxtimes f = f = f \boxtimes 0_{\sqcup} \) and \( 0_{\sqcup} \boxtimes (\sqcap)f = 0_{\sqcup} \boxtimes f = f(\sqcap)0_{\sqcup} = f \).

(5) \( f \boxtimes g = g \boxtimes f \). If \( T \) is commutative, then \( f \boxtimes g = g \boxtimes f \).

Proof. We only show \( (f \boxtimes g) \boxtimes h = f \boxtimes (g \boxtimes h) \). The other properties can be similarly proved. If there does not exist \( y, z \in R \) such that \( x \leq y \cap z \), then

\[
((f \boxtimes g) \boxtimes h)(x) = 0_{R} = (f \boxtimes (g \boxtimes h))(x).
\]

Otherwise, we have

\[
((f \boxtimes g) \boxtimes h)(x) = \bigvee_{x \leq y \cap z} (f \boxtimes g)(y)h(z)
\]

\[
= \left\{ \begin{array}{ll}
\bigvee_{x \leq y \cap z} \left( \bigvee_{y \leq u \cap v} f(u)g(v)h(z) \right) & \text{if there exist } u, v \in T \text{ such that } y \leq u \cap v, \text{ otherwise} \\
0_{R} & \text{otherwise}
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
f(u)g(v)h(z) & \text{if there exist } u, v, z \in T \text{ such that } x \leq u \cap v \cap z, \text{ otherwise} \\
0_{R} & \text{otherwise}
\end{array} \right.
\]

\[
= (f \boxtimes (g \boxtimes h))(x).
\]

It follows \( (f \boxtimes g) \boxtimes h = f \boxtimes (g \boxtimes h) \). \( \Box \)

Lemma 3. Let \( f, g, h \in \mathbb{R}^T \). Then

(1) \( f \boxtimes (g \boxtimes h) \subseteq f \boxtimes g \boxtimes f \boxtimes h \) and \( (g \boxtimes h) \boxtimes f \subseteq g \boxtimes f \boxtimes h \)

(2) \( f(\sqcap)(g \boxtimes h) \subseteq f(\sqcap)g \boxtimes f(\sqcap)h \) and \( (g \boxtimes h)(\sqcap)f \subseteq g(\sqcap)f \boxtimes h(\sqcap)f \).

Proof. We only show (1). (2) can be similarly proved. Let \( x \in T \). If there does not exist \( y, z \in R \) such that \( x \leq y \cap z \), then

\[
(f \boxtimes (g \boxtimes h))(x) = 0_{R} \leq (f \boxtimes g \boxtimes f \boxtimes h)(x).
\]

Otherwise, we have

\[
(f \boxtimes (g \boxtimes h))(x) = \bigvee_{x \leq y \cap z} f(y)g(z)
\]

\[
= \bigvee_{x \leq y \cap z} \left( \bigvee_{y \leq u \cap v} f(y)g(u)h(v) \right)
\]

\[
\leq \bigvee_{x \leq y \cap z} (f(y)g(u))(f(y)h(v))
\]

\[
\leq \bigvee_{x \leq y \cap z} (f \boxtimes g)(y \cap u)(f \boxtimes g)(y \cap v)
\]

\[
\leq \bigvee_{x \leq y \cap z} (f \boxtimes h)(a)(g \boxtimes h)(b)
\]

\[
= (f \boxtimes g \boxtimes f \boxtimes h)(x).
\]

It follows \( f \boxtimes (g \boxtimes h) \subseteq f \boxtimes g \boxtimes f \boxtimes h \). In a similar way we may prove that \( (g \boxtimes h) \boxtimes f \subseteq g \boxtimes f \boxtimes h \). \( \Box \)

Lemma 4. Let \( f, g, h \in \mathbb{R}^T \) be such that \( f \preceq g \). Then

(1) \( f \boxplus h \preceq g \boxplus h \).

(2) \( f \boxtimes h \preceq g \boxtimes h \) and \( f \boxtimes h \preceq h \boxtimes g \).

(3) \( f(\sqcap)h \subseteq g(\sqcap)h \) and \( h(\sqcap)f \subseteq h(\sqcap)(\sqcap)g \).

Proof. We only show (1). (2) and (3) can be similarly proved. Let \( x \in T \). Then

\[
(f \boxplus h)(x) = \bigvee_{x \leq y \cap z} f(y)h(z)
\]

\[
\leq \bigvee_{x \leq y \cap z} g(y)h(z) = (g \boxplus h)(x).
\]

It follows \( f \boxplus h \preceq g \boxplus h \). \( \Box \)

Next, we introduce the concept of \( R \)-valued ideals of \( T \) as follows.

Definition 5. Let \( f \in \mathbb{R}^T \). \( f \) is called an \( R \)-valued ideal of \( T \) if it satisfies the following conditions: \( \forall x, y \in T \)

(1a) \( f(0_T) = 1_R \);

(2a) \( f(x \cap y) \geq f(x)f(y) \);

(3a) \( f(x \cap y) \geq f(x) + f(y) \);

(4a) \( f(x \cap y) \geq 0 \).
Denote by $RT$ the set of all $R$-value ideals of $T$.

**Example 6.** Define on the set $T = \{0, x, y, z\}$ a totally order relation by $0 < x < y < z$ and an addition operation "⊕", a multiplication operation "⊙" by the tables:

<table>
<thead>
<tr>
<th>⊕</th>
<th>0</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>x</td>
<td>y</td>
<td>z</td>
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<td>x</td>
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<table>
<thead>
<tr>
<th>⊗</th>
<th>0</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
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<td>0</td>
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<td>z</td>
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<td>z</td>
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</table>

Then $(T, ⊕, ⊗, ⪯)$ is a positive totally ordered hemiring.

Let $(R, V, A, 0, 1)$ be a complete distributively bounded lattice which is clearly a CLO-semiring if we define $r + s = r \lor s$ and $rs = r \land s$ for all $r, s \in R$. Define $f$ be the function from $T$ into $R$ as follows:

$$f(0) = 1, f(x) = r, f(y) = s \text{ and } f(z) = t,$$

where $0 \leq s < r \leq 1$. Then $f$ is an $R$-valued ideal of $T$.

Let $f \in RT$ and $r \in R$. Define a set $f_r = \{x \in T \mid f(x) > r\}$, called the $r$-strong cut set of $f$. Then it is not difficult to see that the following results are valid, which present the relationships between crisp ideals and $R$-valued ideals of $T$.

**Theorem 7.** Let $A \subseteq T$. Then $A$ is an ideal of $T$ if and only if $\chi_A$ is an $R$-valued ideal of $T$.

**Theorem 8.** Let $f \in RT$. Then $f$ is an $R$-valued ideal of $T$ if and only if $f_t = \{f \in RT \mid f_t \neq 0\}$ is an ideal of $T$ for all $t \in R \setminus \{1\}$.

**Theorem 9.** Let $f \in RT$. Then $\hat{f}$ is an ideal of $T$, where $\hat{f} = \{x \in T \mid f(x) = 1\}$.

**Lemma 10.** Let $f \in RT$ be such that $x \leq y$ implies $f(x) \geq f(y)$. Then (F2a) holds if and only if the following condition holds:

(F2b) $f \oplus f = f$.

**Proof.** (F2a)⇒(F2b) Let $x \in T$. If $x \leq y \oplus z$ for some $y, z \in T$, then $f(x) \geq f(y \oplus z)$. Hence

$$f(x) = \bigvee_{x \leq y \oplus z} f(y)f(z) \leq \bigvee_{x \leq y \oplus z} f(y \oplus z) \leq \bigvee_{x \leq y \oplus z} f(x) = f(x)$$

It follows $f \oplus f \subseteq f$.

On the other hand,

$$f(x) = \bigvee_{x \leq y \oplus z} f(y)f(z) \geq f(x)f(0) = f(x)1_R \Rightarrow f(x) = f(x).$$

It follows $f \subset f \oplus f$. Hence $f \oplus f = f$.

(F2b)⇒(F2a) Let $x, y \in T$. Then

$$f(x \oplus y) = (f \oplus f)(x \oplus y) = \bigvee_{x \oplus y \leq u \oplus v} f(u)f(v) \geq f(x)f(y).$$

Hence (F2a) holds.

**Lemma 11.** Let $f \in RT$ be such that $x \leq y$ implies $f(x) > f(y)$. Then (F3a) holds if and only if the following condition holds:

(F3b) $\chi_T f \subseteq f$ and $f \cap \chi_T \subseteq f$.

**Proof.** The proof is analogous to that of Lemma 7.

**Lemma 12.** Let $f, g \in RT$. Then $f \subseteq f \oplus g$ and $g \subseteq f \oplus g$.

**Proof.** Let $x \in T$. Then

$$(f \oplus g)(x) = \bigvee_{x \leq y \oplus z} f(y)g(z) \geq f(0)g(x) = 1_Rg(x) = g(x).$$

It follows $f \subseteq f \oplus g$ and then $g \subseteq f \oplus g$.

**Theorem 13.** Let $f, g \in RT$. Then so are $f \oplus g$, $f(\oplus g)$ and $f \cap g$.

**Proof.** We only show that $f(\oplus g)$ is an $R$-valued ideal of $T$. The cases for $f \oplus g$ and $f \cap g$ can be similarly proved.

$$\begin{align*}
(1) \quad (f(\oplus g))(0) &= \bigvee_{0 \leq y \leq z} \prod_{x \in T} f(x)(y)g(z) \geq f(0)g(0) = 1_R.
(2) \quad \text{Let } x, y \in T. \text{ If there does not exist } u_i, v_i \leq T \text{ such that } x \leq u_i \oplus v_i \oplus \cdots \oplus u_m \oplus v_m \text{ or } y \leq u_i \oplus v_i \oplus \cdots \oplus u_m \oplus v_m, \text{ then}
\quad (f(\oplus g)(x))(f(\oplus g))(y) = 0_R \leq (f(\oplus g)(x \oplus y))
\end{align*}$$

Otherwise, we have

$$(f(\oplus g))(x \oplus y) = \bigvee_{x \leq y \oplus z} \prod_{i=1}^m f(u_i)g(v_i) \geq \bigvee_{x \leq y \oplus z} \prod_{i=1}^m f(a_i)g(b_i) \prod_{j=1}^{m_2} f(c_j)g(d_j) = \left(\prod_{x \leq y \oplus z} f(a_i)g(b_i)\right) \left(\prod_{y \leq z} f(c_j)g(d_j)\right) = (f(\oplus g))(x)(f(\oplus g))(y).$$

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Hence, in any case, we have \((f(\Box)g)(x \uplus y) \geq (f(\Box)g)(x)(f(\Box)g)(y)\).

(3) Let \(x, y \in T\). If there does not exist \(u_1, v_1 \in T\) such that \(x \leq u_1 \oplus v_1 \oplus \cdots \oplus u_m \oplus v_m\), then
\[
(f(\Box)g)(x) = 0_R \leq (f(\Box)g)(x \oplus y).
\]

Otherwise, we have
\[
(f(\Box)g)(x \oplus y) \\
\geq \bigvee_{x \leq \sum_{i=1}^{m} u_i \oplus v_i} \prod_{i=1}^{m} f(u_i)g(v_i) \\
\geq \bigvee_{x \leq \sum_{i=1}^{m} u_i \oplus v_i} \prod_{i=1}^{m} f(u_i)g(u_i) = (f(\Box)g)(x).
\]

In a similar way, we may prove that \((f(\Box)g)(x \oplus y) \geq (f(\Box)g)(y)\).

(4) Let \(x \leq y\). Then \(y \leq u_1 \oplus v_1 \oplus \cdots \oplus u_m \oplus v_m\) for some \(u_i, v_i \in T\) gives \(x \leq u_1 \oplus v_1 \oplus \cdots \oplus u_m \oplus v_m\), and so \((f(\Box)g)(x) \geq (f(\Box)g)(y)\).

Summing up the above arguments, \(f(\Box)g \in RI^T\).

Lemma 14. Let \(f, g, h \in RI^T\). Then \(f(\Box)(g \uplus h) = f(\Box)g \uplus f(\Box)h\) and \((g \uplus h) \uplus f(\Box)h = g \uplus f(\Box)(h \uplus f(\Box))\).

Proof. We only present a proof for the first equality. By Lemma 3(2), it remains to show \(f(\Box)(g \uplus h) \geq f(\Box)g \uplus f(\Box)h\). Note first that \(g \uplus h \geq g\) and \(g \uplus h \geq h\), and so \((f(\Box)g \uplus f(\Box)h) \geq f(\Box)g\) and \(f(\Box)g \uplus f(\Box)h) \geq f(\Box)h\) by Lemma 4. In addition, it follows from Theorem 13 that \(f(\Box)(g \uplus h) \in RI^T\). Hence \(f(\Box)(g \uplus h) = f(\Box)(g \uplus h) \uplus f(\Box)(g \uplus h) \geq f(\Box)g \uplus f(\Box)h = f(\Box)g \uplus f(\Box)h\). Therefore, \(f(\Box)(g \uplus h) = f(\Box)g \uplus f(\Box)h\).

Theorem 15. \((RI^T, \uplus, \Box)\) is an additively idempotent hemiring with zero element \(0_{RI^T}\). In addition, if \(T\) is commutative, then so is \(RI^T\).

Proof. The desired result follows from the following facts.

(1) \((RI^T, \uplus)\) is a commutative idempotent monoid with identity element \(0_{RI^T}\) by Lemmas 2, 10 and Theorem 13.

(2) \((RI^T, \Box)\) is a semigroup by Lemma 2 and Theorem 13.

(3) \(\Box\) distributes over \(\uplus\) from either side by Lemma 3.14.

Lemma 16. Let \(f, g \in RI^T\) and \(h \in RI^T\). Then \(f \Box g \subseteq h\) if and only if \(f(\Box)g \subseteq h\).

Proof. It is obvious that \(f(\Box)g \subseteq h\) implies \(f \Box g \subseteq h\) by Lemma 2. Now assume that \(f \Box g \subseteq h\). Let \(x \in T\).

If there does not exist \(u_1, v_1 \in T\) such that \(x \leq u_1 \oplus v_1 \oplus \cdots \oplus u_m \oplus v_m\), then
\[
(f(\Box)g)(x) = 0_R \leq (f(\Box)g)(x \oplus y).
\]

Otherwise, we have
\[
(f(\Box)g)(x) = \bigvee_{x \leq \sum_{i=1}^{m} u_i \oplus v_i} \prod_{i=1}^{m} f(u_i)g(v_i) \\
\leq \bigvee_{x \leq \sum_{i=1}^{m} u_i \oplus v_i} \prod_{i=1}^{m} (f \Box g)(u_i \oplus v_i) \\
\leq \bigvee_{x \leq \sum_{i=1}^{m} u_i \oplus v_i} \prod_{i=1}^{m} h(u_i \oplus v_i) \leq h(x).
\]

It follows \(f(\Box)g \subseteq h\).

Theorem 17. \((RI^T, \uplus, \subseteq, \Box)\) is a complete bounded lattice with the minimal element \(0_{RI^T}\) and maximal element \(\chi_T\). In addition, \(f(\Box)g \subseteq f \cap g\) for all \(f, g \in RI^T\).

Proof. Let \(f, g \in RI^T\). It follows from Theorem 13 that \(f \wedge g = f \cap g \in RI^T\) and \(f \uplus g \in RI^T\). We now show that \(f \vee g = f \oplus g\). It is clear that \(f \subseteq f \oplus g\) and \(g \subseteq f \oplus g\). Now, let \(g \in RI^T\) be such that \(f \subseteq h\) and \(g \subseteq h\). Then, \(f \oplus g \subseteq h \uplus g \subseteq g\). Hence \(f \wedge g = f \oplus g\). This implies that \((RI^T, \uplus, \subseteq, \Box)\) is a lattice. And it is easy to see that \(RI^T\) is complete and that \(0_{RI^T}\) and \(\chi_T\) are the minimal element and maximal element of \(RI^T\), respectively. Next, we show that \(f(\Box)g \subseteq f \cap g\). From Lemma 11, \(f \Box g \subseteq f(\Box)g \cap \chi_T\). \(g \subseteq f \cap g\). Hence it follows from Lemma 16 that \(f(\Box)g \subseteq f \cap g\).

From Lemma 4, Theorems 15 and 17, we have the following result.

Theorem 18. \((RI^T, \uplus, \Box)\) is an additively idempotent bounded CLO-hemiring.

THE EMBEDDING OF SOME ORDERED HEMIRINGS

Given \(f \in RI^T\), we call the least \(R\)-valued ideal of \(T\), containing \(f\), the \(R\)-valued ideal of \(T\) generated by \(f\), denoted by \(\langle f \rangle\). By Theorem 13, the \(R\)-valued ideal \(\langle f \rangle\) of \(T\) is actually the intersection \(\bigcap \{g \in RI^T \mid f \subseteq g\}\). From Theorems 15 and 17, it is easy to check that \(f(\Box)g = \langle f \Box g \rangle\) for all \(f, g \in RI^T\).

Lemma 19. Let \(x \in T\) and \(r \in R \setminus \{0_R\}\). Then \((x_r)\) is the \(R\)-valued ideal generated by \(x_r\), where \((x_r)\) is defined by
\[
(x_r)(y) = \begin{cases} 
1_R & \text{if } y = 0_R, \\
r & \text{if } y \in I(x) \setminus \{0_T\}, \\
0_R & \text{otherwise}.
\end{cases}
\]

for all \(y \in T\).
Proof. It is easy to see that \((x_r)\) is an \(R\)-valued ideal of \(T\). Now let \(f \in R^T\) be such that \(x_r \subseteq f\) and \(y \in T\). We show that \(y \in \{0_R\}\). The verification is as follows.

Case 1. \(y \notin I(x)\). Then \((x_r)(y) = 0_R \leq f(y)\).

Case 2. \(y = 0_R\). Then \(f(y) = 1_R = \langle x_r \rangle(y)\).

Case 3. \(y \in \{0_R\}\). Then there exist \(m \in \{0,1,2, \ldots \}\) and \(a,b,c,d \in T\) such that \(y \leq mx \oplus a \odot x \oplus b \oplus c \odot x \oplus d\) and \(0_R = m\).

\[
f(y) \geq f(x)(a \odot f(x) \odot f(c \odot x \odot d) \geq f(x)f(x)f(x) = f(x)
\]

Thus, in any case, we have \(\langle x_r \rangle \subseteq f\). This completes the proof. \(\square\)

Lemma 20. Let \(x,y \in T\) and \(r \in R\setminus\{0_R\}\). If \(T\) is commutative, then

1. \((x_r) \oplus (y_r) = \langle (x \oplus y)_r \rangle\)
2. \((x_r \circ (y_r)) = \langle (x \circ y)_r \rangle\)

Proof. (1) Let \(z \in T\). We consider the following cases.

Case 1. \(z \notin I(x \oplus y)\). Then \(\langle (x \oplus y)_r \rangle(z) = 0_R\) and \(a \notin I(x)\) or \(b \notin I(y)\) for \(z \leq a \oplus b\). Otherwise, if \(a \notin I(x)\) and \(b \notin I(y)\), then there exist \(m,n \in \{0,1, \ldots \}\) and \(c,d \in T\) such that \(z \leq mx \oplus c \odot x \oplus nx \oplus d \odot x\) and \(z \leq a \oplus b \leq mx \oplus c \odot x \oplus nx \oplus d \odot x = (m+n)x \oplus c \odot x \odot d \odot x\) and \(z \leq a \oplus b \leq mx \oplus c \odot x \oplus nx \oplus d \odot x = (m+n)x \oplus c \odot x \odot d \odot x\), i.e., \(z \leq I(x \oplus y)\), a contradiction. Hence \((x_r)(a) = 0_R\) or \((y_r)(b) = 0_R\) and so

\[
\langle x_r \rangle \oplus \langle y_r \rangle \geq \bigvee_{z \leq a \oplus b} \langle x_r \rangle(a) \langle y_r \rangle(b)
\]

Case 2. \(z = 0_R\). Then \(\langle x \oplus y \rangle_r(z) = 1_R\) and \((x_r) \oplus (y_r) = \bigvee_{z \leq a \oplus b} \langle x_r \rangle(a) \langle y_r \rangle(b) \geq \langle x_r \rangle(0_R) \langle y_r \rangle(0_R) = 1_R\).

Case 3. \(z \in I(x \oplus y)\). Then \(\langle x \oplus y \rangle_r(z) = r\) and there exist \(m \in \{0,1, \ldots \}\) and \(a \in T\) such that \(z \leq mx \oplus a \odot x \oplus (x \oplus y) = (mx \oplus a \odot x) \oplus (my \oplus a \odot y)\). Hence \((x_r) \oplus (y_r)(z) \geq \bigvee_{z \leq a \oplus b} \langle x_r \rangle(a) \langle y_r \rangle(b) \geq \langle x_r \rangle(0_R) \langle y_r \rangle(0_R) = 1_R\). It is obvious that \((x_r) \oplus (y_r)(z) = 1_R\) otherwise \(z = 0_R\). Hence \((x_r) \oplus (y_r)(z) = r\).

Thus, in any case, we have \((x_r) \oplus (y_r)(z) = \langle x \oplus y \rangle_r(z)\). It follows \((x_r) \oplus (y_r) = \langle x \oplus y \rangle_r\).

(2) It is obvious that \(\langle x_r \rangle \circ (y_r) = \langle x_r \odot y_r \rangle = \langle x \odot y \rangle_r\). \(\square\)

Theorem 21. If \(T\) is a commutative positive totally ordered hemiring satisfying the following condition: for any \(x,y \in T\), \(x \neq y\) implies \(I(x) \neq I(y)\), then \((T, \circ, \odot, \leq)\) is embedded in \((R^T, \circ, \odot, \leq)\).

Proof. Let \(r \in R\setminus\{0_R\}\). We consider the mapping \(\varphi : T \rightarrow R^T\) where \(\varphi(x) = \langle x_r \rangle\). Then we have:

1. The mapping \(\varphi\) is well defined. The verification is as follows. Let \(x_1, x_2 \in T\) be such that \(y_1 = y_2\). Then

\[
\langle x_1 \rangle(y_1) = \langle x_2 \rangle(y_2) = \begin{cases} 1_R & \text{if } y_1 = 0_R, \\ r & \text{if } y_1 \in I(x) \setminus \{0_R\}, \\ 0_R & \text{otherwise.} \end{cases}
\]

Let \(x_1, x_2, y \in T\) be such that \(x_1 = x_2\). Then

\[
\langle x_{1\circ}(y) = \langle x_{2\circ}(y) = \begin{cases} 1_R & \text{if } y = 0_R, \\ r & \text{if } y \in I(x) \setminus \{0_R\}, \\ 0_R & \text{otherwise.} \end{cases}
\]

2. \(\varphi\) is a homomorphism. The verification is as follows.

(a) \(\varphi\) is isotonous. Let \(x,y \in T\) be such that \(x \leq y\). For any \(z \in T\), we consider the following cases.

Case 1. \(z = 0_R\). Then \(\langle x \circ y \rangle_r(z) = 1_R \) \(\Rightarrow \langle x \circ y \rangle_r = \langle x \circ y \rangle_r = \langle x \circ y \rangle_r\).

Case 2. \(z \notin I(x)\). Then \(\langle x \circ y \rangle_r(z) = 0_R \) \(\Rightarrow \langle x \circ y \rangle_r \leq \langle x \circ y \rangle_r \).

Case 3. \(z \in I(x) \setminus \{0_R\}\). Then there exist \(m \in \{0,1, \ldots \}\) and \(a \in T\) such that \(z \leq mx \oplus a \odot x \leq my \oplus a \odot y\) and \(z \leq x \oplus y \leq mx \oplus a \odot x \oplus my \oplus a \odot y\), so \((x \circ y)_r(z) = r = \langle y \rangle_r(z)\).

Thus, in any case, we have \(\langle x \circ y \rangle_r \leq \langle x \circ y \rangle_r\), i.e., \(\varphi(x) \leq \varphi(y)\).

(b) \(\varphi(x \odot y) = \varphi(x \odot y) = \varphi(x) \odot \varphi(y)\) and \(\varphi(x \odot y) = \varphi(x) \odot \varphi(y)\). This is straightforward from Lemma 20.

(3) \(\varphi\) is reverse isotone. Let \(x,y \in T\) be such that \(x \leq y\). Suppose if possible, \(x \neq y\), then \(x \neq y\) since \(z \leq x \) is a totally order on \(T\). Analogous to the proof of (a), we have \(\varphi(x) \leq \varphi(y)\) since \(I(x) \neq I(y)\), a contradiction. Hence \(x \leq y\). \(\square\)

Example 22. Define on the set \(T = \{0, x, y\}\) a totally ordered relation by \(0 < x < y\), and an addition operation \(\oplus\) and a multiplication operation \(\odot\) by the tables:

<table>
<thead>
<tr>
<th>(\oplus)</th>
<th>(0)</th>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>(0)</td>
<td>(x)</td>
<td>(y)</td>
</tr>
<tr>
<td>(x)</td>
<td>(x)</td>
<td>(x)</td>
<td>(x)</td>
</tr>
<tr>
<td>(y)</td>
<td>(y)</td>
<td>(x)</td>
<td>(y)</td>
</tr>
</tbody>
</table>

Then \((T, \odot, \circ, \leq)\) is commutative positive totally ordered hemiring satisfying the following condition: for any \(a,b \in T\), \(a \neq b\) implies \(I(a) \neq I(b)\) and can be embedded a commutative additively idempotent bounded CLO-hemiring.

Conclusions

In this paper, we introduced the concept of an \(R\)-valued ideal of an ordered hemiring and provided some of its properties. We further showed that the set of all \(R\)-valued ideals of a positive ordered hemiring forms an additively
idempotent bounded CLO-hemiring and that some ordered hemirings can be embedded in it. Our future work on this topic will focus on studying other $R$-valued ideals of hemirings (semirings) such as $R$-valued $k$-ideals and $R$-valued $h$-ideals, and applying the results to the other algebraic structures.

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