Fuzzy Soft Incline Algebras

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Abstract: We introduce the concept of fuzzy soft incline algebras and investigate some of their properties. We discuss fuzzy soft images and fuzzy soft inverse images of fuzzy soft incline algebras. We introduce the notion of an \((\mathcal{E}, \mathcal{E} \cap \mathcal{Q})\)—fuzzy soft incline algebra which is a generalization of a fuzzy soft incline algebra.

Key words: Fuzzy soft incline algebras, \((\mathcal{E}, \mathcal{E} \cap \mathcal{Q})\)—fuzzy soft incline algebra

INTRODUCTION

Cao et al. [1] introduced the notion of incline algebras in their book: Incline algebra and applications, and was studied by some authors [2-10]. Inclines are a generalization of both Boolean and fuzzy algebras, and a special type of a semiring, and they give a way to combine algebras with ordered structures to express the degree of intensity of binary relations. An incline is a structure which has an associative, commutative addition, and a distributive multiplication such that \(x + x = x, x + xy = x\) for all \(x, y\). It has both a semiring structure and a poset structure. Inclines can also be used to represent automata and other mathematical systems, in optimization theory, to study inequalities for nonnegative matrices of polynomials. Ahn et al. [2] introduced the notion of quotient incline and obtained the structure of incline algebras. They also introduced the notion of prime and maximal ideals in an incline, and studied some relations between them in incline algebras.

Most of the recent mathematical methods meant for formal modelling, reasoning and computing are crisp, accurate and deterministic in nature. But in ground reality, crisp data is not always the part and parcel of the problems encountered in different fields. As a consequence various theories including probability, fuzzy sets, intuitionistic fuzzy sets, vague sets, interval mathematics, rough sets have been evolved in process. But difficulties present in all these theories have been shown by Molodtsov in his paper. The cause of these problems is possibly related to the inadequacy of the parameterization tool of the theories. For handling real life ambiguous situations we need to have methodologies which provide some form or other flexible information processing capacity. In 1999, Molodtsov [11] initiated the concept of soft theory as a new mathematical tool for solving the uncertainties which is free from the above difficulties. He successfully applied the soft set theory into several disciplines including game theory, Riemann integration, measure theory. Applications of soft set theory in real life problems are now catching momentum due to the general nature parametrization expressed by a soft set. Maji et al. [12] gave first practical application of soft sets in decision making problems. They also presented the notion of fuzzy soft sets [13], which has proven to be a very natural and effective extension of Molodtsov’s soft sets. Aygünolgu and Aysun [14] first introduced the concept of fuzzy soft groups and studied several of their properties. Authors like Ali and Shabir [15], Yang [16], Yin and Zhan [17], Akram et al. [18] are the contributors of fuzzy soft algebraic structures. In this paper, we introduce the concept of fuzzy soft incline algebras and investigate some of their properties. We discuss fuzzy soft images and fuzzy soft inverse images of fuzzy soft incline algebras. We introduce the notion of an \((\mathcal{E}, \mathcal{E} \cap \mathcal{Q})\)—fuzzy soft incline algebra which is a generalization of a fuzzy soft incline algebra.

REVIEW OF LITERATURE

An incline algebra \((\mathcal{K})\) with two binary operations denoted by “+” and “∗” satisfying the following axioms for all \(x, y \in \mathcal{K}\),

(a) \(x + y = y + x\),
(b) \(x + (y + z) = (x + y) + z\),
(c) \(x \ast (y \ast z) = (x \ast y) \ast z\),
(d) \(x \ast (y + z) = (x \ast y) + (x \ast z)\),
(e) \((y + z) \ast x = (y \ast x) + (z \ast x)\),
(f) \(x + x = x\),
(g) \(x + (x \ast y) = x\),
(h) \(y + (x \ast y) = y\).

Furthermore, an incline algebra \(\mathcal{K}\) is said to be commutative if \(x \ast y = y \ast x\) for all \(x, y \in \mathcal{K}\). For convenience, we pronounce “+” (resp. “∗”) as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if \(x \ast x = x\) for all \(x \in \mathcal{K}\).
Note that \( x \leq y \Leftrightarrow x + y = y \) for all \( x, y \in K \).

A subicline of an incline \( K \) is a non-empty subset \( M \) of \( K \) which is closed under addition and multiplication. A subicline \( M \) is said to be an ideal of an incline \( K \) if \( x \in M \) and \( y \leq x \) then \( y \in M \). An element 0 in an incline algebra \( K \) is a zero element if \( x + 0 = x = 0 \) and \( x \cdot 0 = 0 \cdot x = 0 \), for any \( x \in K \). By a homomorphism of inclines we shall mean a mapping \( f \) from an incline \( K \) into an incline \( L \) such that \( f(x+y) = f(x) + f(y) \) and \( f(x \cdot y) = f(x) \cdot f(y) \) for all \( x, y \in K \).

**Definition 1.** A fuzzy set \( \mu \) in a set \( X \) of the form

\[
\mu(y) = \begin{cases} 
  t \in (0,1], & \text{if } y=x, \\
  0, & \text{if } y \neq x,
\end{cases}
\]

is said to be a fuzzy point with support \( x \) and value \( t \) and is denoted by \( x_t \). For a fuzzy point \( x_t \) and a fuzzy set \( \mu \) in a set \( X \), Pu and Liu gave meaning to the symbol \( x_t \alpha \mu \), where \( \alpha \in \{ 0, q, \in \} \wedge q \). A fuzzy point \( x_t \) is called belong to a fuzzy set \( \mu \), written as \( x_t \in \mu \), if \( \mu(x) \geq t \). A fuzzy point \( x_t \) is said to be quasicoincident with a fuzzy set \( \mu \), written as \( x_t \alpha \mu \), if \( \mu(x) + t > 1 \). To say that \( x_t \in \vee q \mu \) (resp. \( x_t \in \wedge q \mu \)) means that \( x_t \in \mu \) or \( x_t \alpha q \mu \) (resp. \( x_t \in \mu \) and \( x_t \alpha q \mu \)). \( x_t \alpha \mu \) means that \( x_t \alpha \mu \) does not hold, where \( \alpha \in \{ 0, q \in \} \wedge q \).

**Definition 2.** A fuzzy set \( \mu \) in an incline algebra \( K \) is called a fuzzy incline algebra of \( K \) if

1. \( \mu(x+y) \geq \min(\mu(x), \mu(y)) \),
2. \( \mu(x \cdot y) \geq \min(\mu(x), \mu(y)) \)

for all \( x, y \in G \).

**Definition 3.** A fuzzy set \( \mu \) in \( K \) is called an \( (\epsilon, \in \) \}-fuzzy incline algebra of \( K \) if it satisfies the following conditions:

1. \( x_t \in \mu \rightarrow (e)_t \in \vee q \mu \),
2. \( x_t \in \mu \rightarrow (y)_t \in \vee q \mu \),
3. \( x_t \in \mu \rightarrow (x' \cdot y)_t \in \vee q \mu \)

for all \( x, y \in G, s, t \in (0,1] \).

**Proposition 4.** Let \( K \) be an incline algebra. A fuzzy set \( \mu \) in \( K \) is a fuzzy incline algebra of \( K \) if and only if the following assertions are valid:

1. \( x_t \in \mu \rightarrow (y)_t \in \mu \),
2. \( x_t \in \mu \rightarrow (x \cdot y)_t \in \mu \)

for all \( x, y \in G, s, t \in (0,1] \).

Molodtsov defined the notion of soft set in the following way: Let \( U \) be an initial universe and \( E \) be a set of parameters. Let \( P(U) = I^U \) denotes the power set of \( U \) and let \( A \) be a non-empty subset of \( E \). A pair \( (f, A) \) is called a soft set over \( U \), where \( f \) is a mapping given by \( f : A \rightarrow P(U) \). In other words, a soft set over \( U \) is a parameterized family of subsets of the universe \( U \). For \( \epsilon \in A, f(\epsilon) \) may be considered as the set of \( \epsilon \) – approximate elements of the soft set \( (f, A) \). Clearly, a soft set is not just a subset of \( U \). Maji et al. defined fuzzy soft set in the following way:

A pair \( (f, A) \) is called a fuzzy soft set over \( U \), where \( f \) is a mapping given by \( f : A \rightarrow P(U), P(U) = I^U, I = [0,1] \). In general, for every \( \epsilon \in A \), \( f(\epsilon) \) is a fuzzy set of \( U \) and it is called fuzzy value set of parameter \( x \). The set of all fuzzy soft sets over \( U \) with parameters from \( E \) is called a fuzzy soft class, and it is denote by \( FS(U, E) \).

**Definition 5.** A fuzzy soft set \( (f, A) \) over \( U \) is called a null fuzzy soft set, denoted by \( \Phi \), if \( f(\epsilon) \) is the null fuzzy set \( \Phi \) of \( U \), where \( \Phi(x) = 0 \) for all \( x \in U \). A fuzzy soft set \( (g, A) \) over \( U \) is called a whole fuzzy soft set, denoted by \( \Xi \), if \( g(\epsilon) \) is the whole fuzzy set \( \Xi \) of \( U \), where \( \Xi(x) = 1 \) for all \( x \in U \).

To solve decision making problems based on fuzzy soft sets, Feng et al. [19] introduced the following notion called \( t \)-level soft sets of fuzzy soft sets.

**Definition 6.** Let \( (f, A) \) be a fuzzy soft set over \( U \). For each \( t \in [0,1] \), the set \( (f, A)^t = (f^t, A) \) is called an \( t \)-level soft set of \( (f, A) \), where \( f^t = \{ x \in U | f(\epsilon)(x) \geq t \} \) for all \( \epsilon \in A \). Clearly, \( (f, A)^t \) is a soft set over \( U \).

**Definition 7.** Let \( (f, A) \) and \( (g, B) \) be two fuzzy soft sets over \( U \). We say that \( (f, A) \) is a fuzzy soft subset of \( (g, B) \) and write \( (f, A) \subseteq (g, B) \) if

(i) \( A \subseteq B \),
(ii) \( \forall \epsilon \in A, f(\epsilon) \subseteq g(\epsilon) \).

\( (f, A) \) and \( (g, B) \) are said to be fuzzy soft equal and write \( (f, A) = (g, B) \) if \( (f, A) \subseteq (g, B) \) and \( (g, B) \subseteq (f, A) \).

**Definition 8.** Let \( (f, A) \) and \( (g, B) \) be two fuzzy soft sets over \( U \). Then their extended intersection is a fuzzy soft set denoted by \( (h, C) \), where \( C = A \cup B \) and

\[
h(\epsilon) = \begin{cases} 
  f(\epsilon) & \text{if } \epsilon \in A - B, \\
  g(\epsilon) & \text{if } \epsilon \in B - A, \\
  f(\epsilon) \cap g(\epsilon) & \text{if } \epsilon \in A \cap B, 
\end{cases}
\]

for all \( \epsilon \in C \). This is denoted by \( (h, C) = (f, A) \cap (g, B) \).

**Definition 9.** If \( (f, A) \) and \( (g, B) \) are two fuzzy soft sets over the same universe \( U \) then \( (f, A) \cap (g, B) \) is a fuzzy soft set denoted by \( (f, A) \cap (g, B) \), and is defined by \( (f, A) \cap (g, B) = (h, A \times B) \) where, \( h(a, b) = h(\epsilon) \cap g(\epsilon) \) for all \( (a, b) \in A \times B \). Here \( \cap \) is the operation of fuzzy intersection.

**Definition 10.** Let \( (f, A) \) and \( (g, B) \) be two fuzzy soft sets over \( U \). Then their extended union denoted by

\[
\cup(\epsilon) = \begin{cases} 
  f(\epsilon) & \text{if } \epsilon \in A - B, \\
  g(\epsilon) & \text{if } \epsilon \in B - A, \\
  f(\epsilon) \cup g(\epsilon) & \text{if } \epsilon \in A \cup B, 
\end{cases}
\]

for all \( \epsilon \in C \). This is denoted by \( (h, C) = (f, A) \cup (g, B) \).
(h, C), where C = A ∪ B and
\[
h(\varepsilon) = \begin{cases} 
  f_\varepsilon & \text{if } \varepsilon \in A - B, \\
  g_\varepsilon & \text{if } \varepsilon \in B - A, \\
  f_\varepsilon \cup g_\varepsilon & \text{if } \varepsilon \in A \cap B,
\end{cases}
\]
for all \( \varepsilon \in C \). This is denoted by \((h, C) = (f, A) \cup (g, B)\).

**Definition 11.** Let \((f, A)\) and \((g, B)\) be two fuzzy soft sets over a common universe \(U\) with \(A \cap B \neq \emptyset\), then their restricted intersection is a fuzzy soft set \((h, A \cap B)\) denoted by \((f, A) \cap (g, B) = (h, A \cap B)\) where, \(h(\varepsilon) = f(\varepsilon) \cap g(\varepsilon)\) for all \(\varepsilon \in A \cap B\).

**Definition 12.** Let \((f, A)\) and \((g, B)\) be two fuzzy soft sets over a common universe \(U\) with \(A \cap B \neq \emptyset\), then their restricted union is denoted by \((f, A) \cup (g, B)\) and is defined as \((f, A) \cup (g, B) = (h, C)\) where \(C = A \cap B\) and for all \(\varepsilon \in C\), \(h(\varepsilon) = f(\varepsilon) \cup g(\varepsilon)\).

**Definition 13.** The extended product of two fuzzy soft sets \((f, A)\) and \((g, B)\) over \(U\) is a fuzzy soft set, denoted by \((f \circ g, C)\), where \(C = A \cup B\) and
\[
(f \circ g)(\varepsilon) = \begin{cases} 
  f(\varepsilon) & \text{if } \varepsilon \in A - B, \\
  g(\varepsilon) & \text{if } \varepsilon \in B - A, \\
  f(\varepsilon) \circ g(\varepsilon) & \text{if } \varepsilon \in A \cap B,
\end{cases}
\]
for all \(\varepsilon \in C\). This is denoted by \((f \circ g, C) = (f, A) \circ (g, B)\).

**Definition 14.** If \(A \cap B \neq \emptyset\), then the restricted product \((h, C)\) of two fuzzy soft sets \((f, A)\) and \((g, B)\) over \(U\) is defined as the fuzzy soft set \((h, A \cap B)\) denoted by \((f, A) \circ (g, B)\) where \(h(\varepsilon) = f(\varepsilon) \circ g(\varepsilon)\) for all \(\varepsilon \in A \cap B\). Here \(f(\varepsilon) \circ g(\varepsilon)\) is the product of two fuzzy subsets of \(U\).

**FUZZY SOFT INCLINE ALGEBRAS**

**Definition 15.** Let \((f, A)\) be a fuzzy soft set over \(K\). Then \((f, A)\) is said to be a fuzzy soft incline algebra over \(K\) if
(a) \(f(x + y) \geq \min\{f(x), f(y)\}\),
(b) \(f(x \ast y) \geq \min\{f(x), f(y)\}\) for all \(x, y \in K\).

**Example 16.** Consider the incline algebra \(K = \{a, b, c, d\}\) and we define the sum “+” and product “\(\ast\)” on \(K\) as:

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Let \(A = \{a_1, a_2, a_3\}\) and \(f : A \to \mathcal{P}(K)\) be a set-valued function defined by
\[
f(a_1) = \{(a, 0.7), (b, 0.3), (c, 0.6), (d, 0.3)\},
\]
\[
f(a_2) = \{(a, 0.6), (b, 0.2), (c, 0.5), (d, 0.2)\},
\]
\[
f(a_3) = \{(a, 0.7), (b, 0.1), (c, 0.3), (d, 0.1)\}.
\]
Clearly, \((f, A)\) is fuzzy soft sets over \(K\). By routine calculations, it is easy to see that \(f(x)\) is fuzzy incline algebra for \(x \in A\).

We state the following propositions without their proofs.

**Proposition 17.** Let \((f, A)\) and \((g, B)\) be fuzzy soft incline algebra over \(K\), then \((f, A) \cap (g, B)\) is a fuzzy soft incline algebra over \(K\).

**Proposition 18.** Let \((f, A)\) and \((g, B)\) be fuzzy soft incline algebra over \(K\), then \((f, A) \cup (g, B)\) is a fuzzy soft incline algebra over \(K\).

**Proposition 19.** Let \((f, A)\) and \((g, B)\) be fuzzy soft incline algebra over \(K\), if \(A \cap B = \emptyset\) then \((f, A) \circ (g, B)\) is a fuzzy soft incline algebra of \(K\).

**Theorem 20.** Let \((f, A)\) be a fuzzy soft set over \(K\). \((f, A)\) is a fuzzy soft incline algebra if and only if \((f, A)^t\) is a soft incline algebra over \(K\) for each \(t \in [0, 1]\).

**Proof.** Suppose that \((f, A)\) is a fuzzy soft incline algebra. For each \(t \in [0, 1]\), \(x \in A\) and \(x_0, x_2 \in f^t_x\), then \(f(x_1) \geq t\) and \(f(x_2) \geq t\). From Definition 15, it follows that \(f^t_x\) is a fuzzy incline algebra over \(K\). Thus \(f(x_1 + x_2) \geq \min\{f(x_1), f(x_2)\}\), \(f(x_1 + x_2) \geq t\). This implies that \(x_1 + x_2 \in f^t_x\). Likewise, \(x_1 \ast x_2 \in f^t_x\).

This completes the proof.

**Definition 21.** Let \(\phi : X \to Y\) and \(\psi : A \to B\) be two functions, \(A\) and \(B\) are parametric sets from the crisp sets \(X\) and \(Y\), respectively. Then the pair \((\phi, \psi)\) is called a fuzzy soft function from \(X\) to \(Y\).

**Definition 22.** Let \((f, A)\) and \((g, B)\) be two fuzzy soft sets over \(K_1\) and \(K_2\) respectively and let \((\phi, \psi)\) be a fuzzy soft function from \(K_1\) to \(K_2\).

1. The image of \((f, A)\) under the fuzzy soft function \((\phi, \psi)\), denoted by \((\phi, \psi)(f, A)\), is the fuzzy soft set on \(K_2\) defined by \((\phi, \psi)(f, A) = (\phi(f), \psi(A))\), where for all \(k \in \psi(A), y \in K_2\)

\[
\phi(f)(y) = \begin{cases} 
  \bigvee_{x \in \psi^{-1}(y)} f(x) & \text{if } y \in \psi^{-1}(y), \\
  0 & \text{otherwise.}
\end{cases}
\]
(2) The preimage of \((g, B)\) under the fuzzy soft function \((\phi, \psi)\), denoted by \((\phi, \psi)^{-1}(g, B)\), is the fuzzy soft set over \(K_1\) defined by \((\phi, \psi)^{-1}(g, B) = (\phi^{-1}(g), \psi^{-1}(B))\) where \(\phi^{-1}(g)(x) = g_{\phi(a)}(\phi(x))\), for all \(a \in \psi^{-1}(A)\), for all \(x \in K_1\).

**Definition 23.** Let \((\phi, \psi)\) be a fuzzy soft function from \(K_1\) to \(K_2\). If \(\phi\) is a homomorphism from \(K_1\) to \(K_2\) then \((\phi, \psi)\) is said to be fuzzy soft homomorphism. If \(\phi\) is a isomorphism from \(K_1\) to \(K_2\) and \(\psi\) is one-to-one mapping from \(A\) onto \(B\) then \((\phi, \psi)\) is said to be fuzzy soft isomorphism.

**Theorem 24.** Let \((g, B)\) be a fuzzy soft incline algebra over \(K_2\) and let \((\phi, \psi)\) be a fuzzy soft homomorphism from \(K_1\) to \(K_2\). Then \((\phi, \psi)^{-1}(g, B)\) is a fuzzy soft incline algebra over \(K_1\).

**Proof.** Let \(x_1, x_2 \in K_1\), then
\[
\phi^{-1}(g)(x_1 + x_2) = g_{\phi(e)}(\phi(x_1 + x_2)) = g_{\phi(e)}(\phi(x_1) + \phi(x_2)) \\
\geq \min\{g_{\phi(e)}(\phi(x_1)), g_{\phi(e)}(\phi(x_2))\} \\
= \min\{\phi^{-1}(g)(x_1), \phi^{-1}(g)(x_2)\},
\]
\[
\phi^{-1}(g)(x_1 \ast x_2) = g_{\phi(e)}(\phi(x_1) \ast x_2) = g_{\phi(e)}(\phi(x_1) \ast \phi(x_2)) \\
\geq \min\{g_{\phi(e)}(\phi(x_1)), g_{\phi(e)}(\phi(x_2))\} \\
= \min\{\phi^{-1}(g)(x_1), \phi^{-1}(g)(x_2)\},
\]
Hence \((\phi, \psi)^{-1}(g, B)\) is a fuzzy soft incline algebra over \(K_1\).

**Remark 25.** Let \((f, A)\) be a fuzzy soft incline algebra over \(K_1\) and let \((\phi, \psi)\) be a fuzzy soft homomorphism from \(K_1\) to \(K_2\). Then \((\phi, \psi)(f, A)\) may not be a fuzzy soft incline algebra over \(K_2\).

**Fuzzy Soft Incline Algebras**

**Definition 26.** Given a fuzzy set \(\mu\) in \(K\) and \(A \subseteq [0, 1]\), we define two set-valued functions \(f : A \to P(K)\) and \(f_{\mu} : A \to P(K)\) by
\[
f(t) = \{x \in G : x \in A\}, \quad f_{\mu}(t) = \{x \in G : x \in A \land x \in \mu\}
\]
for all \(t \in A\), respectively. Then \((f, A)\) and \((f_{\mu}, A)\) are soft sets over \(K\), which are called an \(\varepsilon\)-soft set and a \(\mu\)-soft set over \(K\), respectively.

**Example 27.** Consider the incline algebra \(K = \{\{0, 0, b, 1\}\}\) and we have define the sum “+” and product “\(\ast\)” on \(K\) as:

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Let \(\mu\) be a fuzzy set in \(G\) defined by \(\mu(0) = 0.7, \mu(a) = 0.8, \mu(b) = 0.8, \mu(1) = 0.4\). Then \(\mu\) is an \((\varepsilon, \mu)\)-fuzzy incline algebra of \(K\). Take \(A = (0, 0.5)\) and let \((f, A)\) be an \(\varepsilon\)-soft set over \(K\). Then
\[
(t) f(t) = G \text{ if } t \\
(2) f(t) = \{0, a, b\} \text{ if } t \in \{0.4, 0.6\}
\]
which are incline algebras of \(K\). Hence \((f, A)\) is a soft incline algebra over \(K\).

**Definition 28.** Let \((f_{\mu}, A)\) be a fuzzy soft set over \(K\). Then \((f_{\mu}, A)\) is said to be an \((\varepsilon, \mu)\)-fuzzy soft incline algebra over \(K\) if \(f_{\mu}(x)\) is an \((\varepsilon, \mu)\)-fuzzy soft incline algebra of \(K\) for all \(x \in A\).

**Example 29.** Consider the incline algebra \(K = \{0, a, b, 1\}\) and \(+\) and \(\ast\) are given by the Cayley’s table in Example 27. Let \(A = \{e_1, e_2\}\) and \(f_{\mu} : A \to P(K)\) be a set-valued function defined by
\[
f_{\mu}(e_1) = \{(0, \mu(0)), (a, \mu(a)), (b, \mu(b)), (1, \mu(1))\},
\]
\[
f_{\mu}(e_2) = \{(0, 0.7), (a, 0.4), (b, 0.6), (1, 0.4)\},
\]
\[
f_{\mu}(e_2) = \{(0, 0.8), (a, 0.5), (b, 0.7), (1, 0.5)\}.
\]
Clearly, \((f_{\mu}, A)\) is fuzzy soft set over \(K\). Since \(\mu(x + y) \geq \min(\mu(x), \mu(y), 0.5), \mu(x \ast y) \geq \min(\mu(x), \mu(y), 0.5)\) hold for all \(x, y \in K\), \(f_{\mu}(x)\) is an \((\varepsilon, \mu)\)-fuzzy soft incline algebra of \(K\) for all \(x \in A\). Hence \((f_{\mu}, A)\) is an \((\varepsilon, \mu)\)-fuzzy soft incline algebra over \(K\).

The proofs of the following propositions are obvious.

**Proposition 30.** Let \((f_{\mu}, A)\) and \((g_{\mu}, B)\) be \((\varepsilon, \mu)\)-fuzzy soft incline algebras over \(K\), then \((f_{\mu}, A) \cup (g_{\mu}, B)\) is an \((\varepsilon, \mu)\)-fuzzy soft incline algebra over \(K\).

**Proposition 31.** Let \((f_{\mu}, A)\) and \((g_{\mu}, B)\) be \((\varepsilon, \mu)\)-fuzzy soft incline algebras over \(K\), then \((f_{\mu}, A) \land (g_{\mu}, B)\) is an \((\varepsilon, \mu)\)-fuzzy soft incline algebra over \(K\).

**Proposition 32.** Let \((f_{\mu}, A)\) and \((g_{\mu}, B)\) be \((\varepsilon, \mu)\)-fuzzy soft incline algebras over \(K\). If \(A \cap B = \emptyset\) then \((f_{\mu}, A) \cup (g_{\mu}, B)\) is an \((\varepsilon, \mu)\)-fuzzy soft incline algebra of \(K\).

**Proposition 33.** Let \(\mu\) be a fuzzy set in a \(K\)-algebra \(K\) and let \((f, A)\) be an \(\varepsilon\)-soft set on \(K\) with \(A = (0, 1)\). Then \((f, A)\) is a soft incline algebra on \(K\) if and only if \(\mu\) is a fuzzy incline algebra of \(K\).

**Proof.** Assume that \((f, A)\) is a soft incline algebra on \(K\). If \(\mu\) is not a fuzzy incline algebra of \(K\), then there exist \(a, b \in K\) such that \(\mu(a + b) < \min(\mu(a), \mu(b))\). Take \(t \in A\) such that \(\mu(a + b) < t \leq \min(\mu(a), \mu(b))\). Then \(a_{\mu} \in \mu\) and \(b_{\mu} \in \mu\) but \((a + b)_{\min(t, t')} = (a + b) \notin \mu\). Hence \(a, b \in f(t), a + b \notin f(t), a\) contradiction. Thus, \(\mu(x + y) \geq \min(\mu(x), \mu(y))\) for all \(x, y \in K\). Likewise, \(\mu(x + y) \geq \min(\mu(x), \mu(y))\).
Conversely, suppose that $\mu$ is a fuzzy incline algebra of $\mathcal{K}$. Let $t \in \mu$ and $x, y \in f(t)$. Then $x_t$ and $y_t \in \mu$. It follows from Proposition 4 that $(x + y)_t = (x + y)_{\min(t, t)} \in \mu$ so that $x + y \in f(t)$. Likewise, $x \ast y \in f(t)$. Hence $f(t)$ is an incline algebra of $K$, i.e., $(f, A)$ is a soft incline algebra on $\mathcal{K}$.

**Proposition 34.** Let $\mu$ be a fuzzy set in an incline algebra $\mathcal{K}$ and let $(f, A)$ be a $q$-soft set over $\mathcal{K}$ with $A = (0, 1]$. Then $(f_\mu, A)$ is a soft incline algebra over $\mathcal{K}$ if and only if $\mu$ is a fuzzy incline algebra of $\mathcal{K}$.

**Proof.** Suppose that $\mu$ is a fuzzy incline algebra of $\mathcal{K}$. Let $t \in A$ and $x, y \in f_\mu(t)$. Then $x_t \mu$ and $y_t \mu$, i.e., $\mu(x) + t > 1$ and $\mu(y) + t > 1$. It follows from Definition 2 that $\mu(x + y) + t \geq \min(\mu(x), \mu(y)) + t = \min(\mu(x) + t, \mu(y) + t) > 1$ so that $(x + y)_t \mu$, i.e., $x + y \in f_\mu(t)$. Likewise, $x \ast y \in f_\mu(t)$. Hence $f_\mu(t)$ is an incline algebra of $\mathcal{K}$ for all $t \in A$, and so $(f_\mu, A)$ is a soft incline algebra over $\mathcal{K}$. The proof of converse part is obvious. This complete the proof.

**Proposition 35.** Let $\mu$ be a fuzzy set in an incline algebra $\mathcal{K}$ and let $(f, A)$ be an $\varepsilon$-soft set on $\mathcal{K}$ with $A = (0, 1]$. Then the following assertions are equivalent:

(a) $(f, A)$ is a soft incline algebra over $\mathcal{K}$,

(b) $\max(\mu(x + y), 0.5) \geq \min(\mu(x), \mu(y))$,

(c) $\max(\mu(x \ast y), 0.5) \geq \min(\mu(x), \mu(y))$

for all $x, y \in \mathcal{K}$.

**Lemma 36.** A fuzzy set $\mu$ in an incline algebra $\mathcal{K}$ is an $(\varepsilon, \varphi)$-fuzzy incline algebra of $\mathcal{K}$ if and only if it satisfies:

- $\mu(x + y) \geq \min(\mu(x), \mu(y), 0.5)$,

- $\mu(x \ast y) \geq \min(\mu(x), \mu(y), 0.5)$

for all $x, y \in \mathcal{K}$.

**Proposition 37.** Let $\mu$ be a fuzzy set in an incline algebra $\mathcal{K}$ and let $(f, A)$ be an $\varepsilon$-soft set on $\mathcal{K}$ with $A = (0, 0.5]$. Then the following assertions are equivalent:

(i) $\mu$ is an $(\varepsilon, \varphi)$-fuzzy incline algebra of $\mathcal{K}$,

(ii) $(f, A)$ is a soft incline algebra over $\mathcal{K}$.

**Proof.** Assume that $\mu$ is an $(\varepsilon, \varphi)$-fuzzy incline algebra of $\mathcal{K}$. Let $t \in A$ and $x, y \in f(t)$. Then $x_t \in \mu$ and $y_t \in \mu$ or equivalently $\mu(x) \geq t$ and $\mu(y) \geq t$. It follows from Lemma 6 that $\mu(x + y) \geq \min(\mu(x), \mu(y), 0.5) \geq \min(t, 0.5) = t$, so that $(x + y)_t \in \mu$, or equivalently $x + y \in f(t)$. Likewise, $x \ast y \in f(t)$. Hence $(f, A)$ is a soft incline algebra on $\mathcal{K}$.

Conversely, suppose that (ii) is valid. If there exist $a, b \in G$ such that $\mu(a + b) < \min(\mu(a), \mu(b), 0.5)$, then we take $t \in (0, 1)$ such that $\mu(a + b) < t \leq \min(\mu(a), \mu(b), 0.5)$.

Thus $t \leq 0.5, a_t \mu$ and $b_t \mu$, i.e., $a \in f(t)$ and $b \in f(t)$. Since $f(t)$ is a incline algebra of $\mathcal{K}$, it follows that $a + b \in f(t)$ for all $t \leq 0.5$ so that $(a + b)_t \in \mu$ or equivalently $\mu(a + b) \geq t$ for all $t \leq 0.5$, a contradiction. Hence

$\mu(x + y) \geq \min(\mu(x), \mu(y), 0.5)$ for all $x, y \in \mathcal{K}$.

Likewise,

$\mu(x \ast y) \geq \min(\mu(x), \mu(y), 0.5)$ for all $x, y \in \mathcal{K}$.

It follows from Lemma 36 that $\mu$ is an $(\varepsilon, \varphi)$-fuzzy incline algebra of $\mathcal{K}$.

We state the following proposition without its proof.

**Proposition 38.** Let $H$ be an incline algebra of an incline algebra $\mathcal{K}$ and let $(f, A)$ be a soft set over $\mathcal{K}$. If $A = (0, 0.5]$, then there exists an $(\varepsilon, \varphi)$-fuzzy incline algebra $\mu$ of $\mathcal{K}$ such that

$f(t) = \{x \in H \mid x_t \in \mu\}$

$= H \forall \ t \in A$.

**Proposition 39.** Let $\mu$ be a fuzzy set in an incline algebra $\mathcal{K}$ and let $(f_\mu, A)$ be a $q$-soft set over $\mathcal{K}$ with $A = (0, 1]$. Then the following assertions are equivalent:

(i) $\mu$ is a fuzzy incline algebra of $\mathcal{K}$,

(ii) $(f_\mu(t) \neq \emptyset \rightarrow f_\mu(t))$ is a incline algebra of $\mathcal{K}$ for all $t \in A$.

**Proof.** Assume that $\mu$ is a fuzzy incline algebra of $\mathcal{K}$. Let $t \in A$ be such that $f_\mu(t) \neq \emptyset$. Let $x, y \in \mathcal{K}$ such that $x + y \in f_\mu(t), e \in f_\mu(t)$. Then $(x + y)_t \mu$ and $x \ast y \in f_\mu(t)$ or equivalently, $\mu(x + y) + t > 1, \mu(x \ast y) + t > 1$.

Using Definition 2, we have

$\mu(x + y) + t \geq \mu(x) + t > 1,$

$\mu(x \ast y) + t \geq \mu(x) + t > 1,$

$\mu(x + y) + t \geq \min(\mu(x), \mu(y)) + t = \min(\mu(x) + t, \mu(y) + t) > 1,$

and $(x + y)_t \mu$ and $x \ast y \in f_\mu(t)$, $x + y \in f_\mu(t)$. Thus $f_\mu(t)$ is an incline algebra of $\mathcal{K}$.

Conversely, assume that (ii) is valid. Suppose there exist $a, b \in G$ such that $\mu(a + b) < \min(\mu(a), \mu(b)).$ Then $\mu(a + b) + s < 1 < \min(\mu(a), \mu(b)) + s$ for some $s \in A$. It follows that $(a)_t \mu$ and $b_t \mu$, i.e., $a \in f_\mu(s)$ and $b \in f_\mu(s)$. Since $f_\mu(s)$ is an incline algebra of $\mathcal{K}$, we get $a + b \in f_\mu(s)$, and so $(a + b)_t \mu$ or equivalently $\mu(a + b) + s > 1$, a contradiction. Thus $\mu(x + y) \geq \min(\mu(x), \mu(y))$ for all $x, y \in \mathcal{K}$. Hence $\mu$ is a fuzzy incline algebra of $\mathcal{K}$.
REFERENCES


