Characterizations of Regular Ordered Semigroups in Terms of New Fuzzy Ideals

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Abstract: We first introduce the concepts of \((\varepsilon, \in \lor q_k)\)-fuzzy left ideals (right ideals, bi-ideals) and \((\tau, \tau \lor \overline{\tau})\)-fuzzy left ideals (right ideals, bi-ideals) of an ordered semigroup and investigate some related properties. We then present several characterizations of regular ordered semigroups in terms of \((\varepsilon, \in \lor q_k)\)-fuzzy left ideals (right ideals, bi-ideals) and \((\tau, \tau \lor \overline{\tau})\)-fuzzy left ideals (right ideals, bi-ideals).

Key words: Ordered semigroups; \((\varepsilon, \in \lor q_k)\)-fuzzy left (resp. right) ideals; \((\varepsilon, \in \lor q_k)\)-fuzzy bi-ideals; \((\tau, \tau \lor \overline{\tau})\)-fuzzy left (resp. right) ideals; \((\tau, \tau \lor \overline{\tau})\)-fuzzy bi-ideals

INTRODUCTION

Fuzzy set theory, which was founded by Zadeh in 1965, has emerged as a powerful way of representing quantitatively and manipulating the imprecision in decision-making problems. Fuzzy sets or fuzzy numbers can appropriately represent imprecise parameters, and can be manipulated through different operations on fuzzy sets or fuzzy numbers. Since imprecise parameters are treated as imprecise values instead of precise ones, the process will be more powerful and its results more credible. Fuzzy set theory has been studied extensively over the past 45 years. Fuzzy set theory is now applied to problems in engineering, business, medical and related health sciences, and the natural sciences. In an effort to gain a better understanding of the use of fuzzy set theory in industrial engineering (IE) and to provide a basis for future research, a literature review of fuzzy set theory in IE has been conducted. Over the years there have been successful applications and implementations of fuzzy set theory in IE. Fuzzy set theory is being recognized as an important problem modeling and solution technique. The use of fuzzy set theory as a methodology for modeling and analyzing decision systems is of particular interest to researchers in IE due to fuzzy set theory’s ability to quantitatively and qualitatively model problems which involve vagueness and imprecision. Fuzzy set theory can be used to bridge modeling gaps in descriptive and prescriptive decision models in IE.

Murali [1] proposed a definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on a fuzzy set. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [2] played a vital role to generate some different types of fuzzy subgroups. A new type of fuzzy subgroups, \((\varepsilon, \in \lor q)\)-fuzzy subgroups, was introduced in earlier paper Bhakat and Das [3] by using the combined notions of belongingness and quasi-coincidence of fuzzy point and fuzzy set. In fact, \((\varepsilon, \in \lor q)\)-fuzzy subgroup is an important and useful generalization of Rosenfeld’s fuzzy subgroup [4]. After this, Davvaz, Corsini, Jun, Yin and Zhan applied this concept to different algebraic structures (for example, see [5-15]. Recently, Jun et al. [16] introduced the concept of \((\varepsilon, \in \lor q)\)-fuzzy bi-ideals of an ordered semigroup and discussed some properties. In this paper, using the idea of quasi-coincidence of a fuzzy point with a fuzzy set, two new ordering relation \(\subset \lor q_k\) and \(\tau \lor \overline{\tau}\) on the set of all fuzzy subsets of an ordered semigroup are defined. Based on the new ordering relations, we introduce and investigate the concepts of \((\varepsilon, \in \lor q_k)\)-fuzzy left ideals (right ideals, bi-ideals)
\((\exists, \exists \lor \exists \pi)\)-fuzzy left ideals (right ideals, bi-ideals) of an ordered semigroup. We also represent several characterizations of regular ordered semigroups in terms of \((\exists, \exists \lor \exists \pi)\)-fuzzy left ideals (right ideals, bi-ideals) and \((\exists, \exists \lor \exists \pi)\)-fuzzy left ideals (right ideals, bi-ideals).

Preliminaries

In this Section we summarize some basic concepts (see [17-20]) about ordered semigroups that will be used throughout the paper.

An ordered semigroup is an algebraic system \((S, +, \leq)\) consisting of a non-empty set \(S\) together with a binary operation \(+\) and a compatible ordering \(\leq\) on \(S\) such that \((S, +)\) is a semigroup and \(x \leq y\) implies \(ax \leq ay\) and \(xa \leq ya\) for all \(x, y, a \in S\).

Let \((S, +, \leq)\) be an ordered semigroup. A subset \(A\) of \(S\) is called a right (resp. left) ideal of \(S\) if

1. \(AS \subseteq A\) (resp. \(SA \subseteq A\))
2. If \(a \in A\) and \(S \ni b \leq a\), then \(b \in A\).

If \(A\) is both a right and a left ideal of \(S\), then \(A\) is called an ideal of \(S\). Moreover, a subset \(B\) of \(S\) is called a bi-ideal if

1. \(BB \subseteq B\)
2. \(BSB \subseteq B\)
3. If \(a \in B\) and \(S \ni b \leq a\), then \(b \in B\).

For \(A \subseteq S\), we denote \(\langle A \rangle := \{t \in S | t \leq h\text{ for some } h \in A\}\).

Lemma 0.1. For an ordered semigroup \(S\), we have

1. \(\langle A \rangle \subseteq \langle A \rangle\) \(\forall A \subseteq S\).
2. If \(A \subseteq B \subseteq S\), then \(\langle A \rangle \subseteq \langle B \rangle\).
3. \(\langle A \rangle \cap \langle B \rangle = \langle A \cap B \rangle\) \(\forall A, B \subseteq S\).
4. \(\langle A \rangle[B] = \langle AB \rangle\) \(\forall A, B \subseteq S\).
5. For any left (resp. right) ideal or bi-ideal \(A\) of \(S\), we have \(\langle A \rangle\).

A fuzzy subset of an ordered semigroup \(S\), by definition, an arbitrary mapping \(f : S \rightarrow [0, 1]\), where \([0, 1]\) is the usual interval of real numbers. The set of all fuzzy subsets of \(S\) is denoted by \(\mathcal{F}(S)\). For any \(A \subseteq S\), characteristic function \(f_A : S \rightarrow [0, 1]\) is defined by

\[f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}\]

for all \(x \in S\). A fuzzy subset \(f\) in \(S\) defined by

\[f(y) = \begin{cases} r(\neq 0) & \text{if } y = x, \\ 0 & \text{otherwise} \end{cases}\]

is said to be a fuzzy point with support \(x\) and value \(r\) and is denoted by \(x_r\).

In what follows let \(k\) denote an arbitrary element of \([0, 1]\) unless otherwise specified. For a fuzzy point \(x_r\) and a fuzzy subset \(f\) in \(S\), we say that

1. \(x_r \in f\) if \(f(x) \geq r\).
2. \(x_r \in f\) if \(f(x) + r + k > 1\).
3. \(x_r \not\in f\) if \(f(x) < r\).
4. \(x_r \not\in f\) if \(x \not\in f\) or \(x \not\in f\).

Next we define two new ordering relations \(\subseteq \lor \exists \pi\) and \(\subseteq \lor \exists \pi\) on \(\mathcal{F}(S)\) as follows:

1. For any \(f, g \in \mathcal{F}(S)\), by \(f \subseteq \lor \exists \pi g\) we mean that \(x_r \in f \Rightarrow x_r \in \exists \pi g\) for all \(x \in S\) and \(r \in [0, 1]\).
2. For any \(f, g \in \mathcal{F}(S)\), by \(f \subseteq \lor \exists \pi g\) we mean that \(x_r \in f \Rightarrow x_r \in \exists \pi g\) for all \(x \in S\) and \(r \in [0, 1]\).

In the sequel, unless otherwise stated, \(S\) will represent an ordered semigroup, \(\exists \alpha\) means \(\alpha\) does not hold, where \(\alpha \in \{\in \exists \pi, \exists \lor \exists \pi, \exists \exists \pi, \exists \forall \exists \pi\}\).

The proofs of the following Lemmas are obvious.

Lemma 0.2. For any fuzzy subsets \(f, g\) of \(S\),

1. \(f \subseteq \exists \pi g\) if and only if \(g(x) \geq \min\{f(x), \frac{1-k}{2}\}\) for all \(x \in S\).
2. \(f \lor \exists \pi g\) if and only if \(\max\{f(x), \frac{1-k}{2}\} \geq g(x)\) for all \(x \in S\).

Lemma 0.3. For any fuzzy subsets \(f, g\) of \(S\),

1. \(f \subseteq \exists \forall g\) if and only if \(g \subseteq \exists \pi g\).
2. \(f \lor \exists \forall g\) if and only if \(\exists \pi g\).

Now define two relations \(\leq_{\mathcal{F}}\) and \(\geq_{\mathcal{F}}\) on \(\mathcal{F}(S)\) by

\[f \leq_{\mathcal{F}} g\] if and only if \(f \subseteq \exists \pi g\) and \(g \subseteq \exists \pi f\) for all \(f, g \in \mathcal{F}(S)\),

and

\[f \geq_{\mathcal{F}} g\] if and only if \(f \lor \exists \forall g\) and \(g \lor \exists \forall f\) for all \(f, g \in \mathcal{F}(S)\).

Lemma 2.3 gives that \(\leq_{\mathcal{F}}\) and \(\geq_{\mathcal{F}}\) are equivalence relations on \(\mathcal{F}(S)\).

For \(x \in S\), define \(A_x := \{(y, z) \in S \times S | x \leq yz\}\). For two fuzzy subsets \(f, g\) of \(S\), define \(A = f \circ g : S \rightarrow [0, 1]\) \(a \rightarrow \min\{f(y), g(z)\}\) if \(A_x \neq \emptyset\),

and \(0\) otherwise.

One can easily see that the multiplication \(\circ\) on \(\mathcal{F}(S)\) is well defined and associative. Let \(f, g\) be any fuzzy subsets of \(S\). Define the fuzzy subsets \(1\), \(f \lor g\) and \(f \land g\) of \(S\) as follows:

\[1 : S \rightarrow [0, 1]|a \rightarrow 1(a) := 1\]

\[f \lor g : S \rightarrow [0, 1]|a \rightarrow (f \lor g)(a) := \max\{f(a), g(a)\}\]

and

\[f \land g : S \rightarrow [0, 1]|a \rightarrow (f \land g)(a) := \min\{f(a), g(a)\}\]

Lemma 0.4. Let \(f_1, f_2, g_1\) and \(g_2\) be any fuzzy subsets of \(S\).

1. If \(f_1 \subseteq \exists \forall g_2, f_1 \subseteq \exists \pi g_2, g_2 \subseteq \exists \pi f_2\), then \(f_1 \circ g_1 \subseteq \exists \forall f_2 \circ g_2\).
2. If \(f_1 \subseteq \forall \exists \pi f_2\), then \(f_1 \circ g_1 \subseteq \forall \exists \pi f_2 \circ g_2\).
Proof. It is straightforward. □

Lemma 0.5. Let A, B ⊆ S. Then we have
(1) A ⊆ B ↔ f_A ⊆ ∀B f_B ↔ f_A ≤ ∀B f_B.
(2) f_A ∩ f_B = f_{A∩B}.
(3) f_A o f_B = f_{A∩B}.

Proof. It is straightforward. □

NEW FUZZY IDEALS OF ORDERED SEMIGROUPS

Definition 0.6. A fuzzy subset f of S is called an (ε, ∈\text{null})-fuzzy left ideal if for all x, y ∈ S and r ∈ (0, 1] we have
(F1a) 1 o f ⊆ null f,
(F2a) If x ≤ y, then y_r ∈ f ⇒ x_r ∈ f.

(ε, ∈\text{null})-fuzzy right ideals and (ε, ∈\text{null})-fuzzy ideals are defined similarly.

Example 0.7. Consider a set S = {0, a, b, c} with the following multiplication “·” and order relation “≤”:
\[\begin{array}{cccc}
0 & a & b & c \\
0 & 0 & 0 & 0 \\
a & 0 & 0 & c \\
b & 0 & 0 & c \\
c & 0 & 0 & c \\
\end{array}\]
The pair (S, ≤) is an ordered semigroup. Define a fuzzy subset f of S by
f(0) = 0.4, f(a) = 0.6, f(b) = 0.2 and f(c) = 0.2.

Then f is an (ε, ∈\text{null})-fuzzy ideal of S.

Theorem 0.8. A fuzzy subset f of S is an (ε, ∈\text{null})-fuzzy left (resp. right) ideal of S if and only if it satisfies the following conditions: for all x, y ∈ S,
(F1b) f(xy) ≥ min{f(y), \frac{1-k}{2}} (resp. f(xy) ≥ min{f(x), \frac{1-k}{2}}).
(F2b) If x ≤ y, then f(x) ≥ min{f(y), \frac{1-k}{2}}.

Proof. We only consider the case for (ε, ∈\text{null})-fuzzy left ideals, the case for (ε, ∈\text{null})-fuzzy right ideals is similar.

Let f be an (ε, ∈\text{null})-fuzzy left ideal of S. If there exist x, y ∈ S such that f(xy) < r = min{f(y), \frac{1-k}{2}} then f(xy) + r + k < 2r + k ≤ 2\frac{1-k}{2} + k = 1.
Hence (xy)_r ∈ null f. On the other hand, (1 o f)(xy) = \bigvee \{ f(b) \geq f(y) \geq r, \text{ that is, } (xy)_r \in 1 o f, \text{ a contradiction. Hence (F1b) is satisfied. To prove (F2b) holds, let } x \leq y. \text{ If } f(x) < r = min\{f(y), \frac{1-k}{2}\}, \text{ then } y_r \in f \text{ but } x_r \in null f, \text{ which contradicts } y_r \in f \Rightarrow x_r \in null f. \text{ Thus (F2b) is valid.}

Conversely, assume that the given conditions hold.
If there exists x_r ∈ 1 o f such that x_r \in null f, then f(x) < r and f(x) + r + k ≤ 1. It follows that f(x) < \frac{1-k}{2}. Now, for any (y, z) ∈ A_x, i.e., x ≤ y, z, we have \frac{1}{2+k} > f(x) > \min\{f(yz), \frac{1}{2+k}\} = f(yz) ≥ \min\{f(z), \frac{1}{2+k}\} = f(z), \text{ and so}
r ≤ (1 o f)(x) = \bigvee \{ f(z) ≤ \bigvee \{ f(x) = f(x), \text{ a contradiction. Hence, } x_r \in null f. \text{ This implies that } 1 o f \subseteq null f, \text{ i.e., (F1a) holds. Next, let } x ≤ y \text{ and } y_r \in f. \text{ If } x_r \in null f, \text{ then } f(x) < r \text{ and } f(x) + r + k \leq 1. \text{ Thus, } f(y) ≤ r, f(x) < r \text{ and } f(x) < \frac{1}{2+k}, \text{ which contradicts } f(x) ≥ \min\{f(y), \frac{1}{2+k}\}. \text{ Therefore } x_r \in null f \text{ and so } f \text{ is an } (ε, ∈\text{null})-\text{fuzzy left ideal of } S. □

Theorem 0.9. A fuzzy subset f of S is an (ε, ∈\text{null})-fuzzy left (resp. right) ideal of S if and only if for all x, y ∈ S and r ∈ (0, 1], it satisfies (F2a) and
(F1c) y_r ∈ f implies (xy)_r ∈ null f (resp. x_r ∈ f implies (yx)_r ∈ null f).

Proof. We only consider the case for (ε, ∈\text{null})-fuzzy left ideals, the case for (ε, ∈\text{null})-fuzzy right ideals is similar.

Let f be an (ε, ∈\text{null})-fuzzy left ideal of S. If there exist x, y ∈ S and r ∈ (0, 1] such that y_r ∈ f but (xy)_r \in null f, then f(y) > r, f(xy) < r and f(xy) + r + k ≤ 1. It follows that f(xy) < f(y) + 2f(xy) + k < f(xy) + r + k ≤ 1, i.e., f(xy) < \frac{1}{2+k}, \text{ which contradicts } f(x) ≥ \min\{f(y), \frac{1}{2+k}\}. \text{ Hence (F1c) is satisfied.}

Conversely, assume that the given conditions hold. If there exist x, y ∈ S such that f(xy) < r = \min\{f(y), \frac{1}{2+k}\}, then y_r ∈ f but (xy)_r \in null f, a contradiction. Hence (F1b) is valid. Therefore, f is an (ε, ∈\text{null})-fuzzy left ideal of S. □

Proposition 0.10. Let A be a non-empty subset of S. Then A is a left (resp. right) ideal of S if and only if the fuzzy subset f of S such that f(x) ≥ \frac{1}{2+k} for all x ∈ A and f(x) = 0 otherwise is an (ε, ∈\text{null})-fuzzy left (resp. right) ideal of S.

Proof. We only consider the case for (ε, ∈\text{null})-fuzzy left ideals, the case for (ε, ∈\text{null})-fuzzy right ideals is similar.

Assume that A is a left ideal of S. Let x, y ∈ S and r ∈ (0, 1] be such that y_r ∈ f. Then f(y) ≤ r > 0 and so f(xy) ≥ \frac{1}{2+k}, i.e., y_r ∈ A. Hence xy ∈ A, which implies that f(xy) ≥ \frac{1}{2+k}. If r ≤ \frac{1}{2+k}, then f(xy) ≥ \frac{1}{2+k} ≥ r > 0, i.e., (xy)_r ∈ f. If r > \frac{1}{2+k}, then f(xy) + r + k > \frac{1}{2+k} + \frac{1}{2+k} + k = 1, i.e., (xy)_r \notin f. Therefore, (xy)_r ∈ null f.

Now let x, y ∈ S and r ∈ (0, 1] be such that y ≤ x and x_r ∈ f. Then f(x) ≥ r > 0, i.e., x ∈ A, and so y ≤ x. If y ≤ x ≤ y_r ∈ f. Then f(y) ≥ r > 0, i.e., y_r ∈ A, and so y ∈ A. If y ≤ x, then f(y) ≥ r > 0, i.e., y_r ∈ f. If r > \frac{1}{2+k}, then f(y) + r + k > \frac{1}{2+k} + \frac{1}{2+k} + k = 1, i.e., y_r \notin f. Therefore, y_r ∈ null f.

Therefore, f is an (ε, ∈\text{null})-fuzzy left ideal of S.
Conversely, assume that \( f \) is an \((\epsilon, \in \vee \mu_k)\)-fuzzy left ideal of \( S \). Let \( x, y, z \in S \) and \( y \neq z \). Then \( f(x, f(y)) \geq \frac{1-k}{2} \) and \( yf(y) = f \). It follows that \( (x, y, f) \vee \mu_k \in f \), i.e., \( (x, y, f) \in f \) or \( (x, y, f) \notin f \). Hence \( f(x, y, f) \geq f(f) \) or \( f(x, y, f) + f(y) + k > 1 \), and so \( f(x, y, f) \geq f(f) \geq \frac{1-k}{2} \) or \( f(x, y, f) + f(y) + k > 1 - \frac{1-k}{2} - k = 1 - \frac{k}{2} \). Thus, in any case \( f(x, y, f) \geq \frac{1-k}{2} \). Hence \( yf(x) \in A \). Similarly, we may show that \( x \leq y \) and \( y \in A \) imply \( x \in A \). Therefore, \( A \) is a left ideal of \( S \). □

**Proposition 0.11.** Let \( A \subseteq S \). Then \( A \) is a left (resp. right) ideal of \( S \) if and only if \( f_A \) is an \((\epsilon, \in \vee \mu_k)\)-fuzzy left (resp. right) ideal of \( S \).

**Proof.** The proof is analogous to that of Proposition 3.5. □

Next, we consider another generalized fuzzy left (resp. right) ideal of \( S \), which is called an \((\epsilon, \in \vee \mu_k)\)-fuzzy left (resp. right) ideal.

**Definition 0.12.** A fuzzy subset \( f \) of \( S \) is called an \((\epsilon, \in \vee \mu_k)\)-fuzzy left ideal if for all \( x, y, z \in S \) and \( r \in [0, 1] \) we have

\[
(F3a) \ f(\epsilon \vee \mu_k, 1 \circ f) \in f,
\]

\[
(F4a) \text{If } x \leq y, \text{ then } x \circ f \Rightarrow y \circ f \cap \in \vee \mu_k f.
\]

\((\epsilon, \in \vee \mu_k)\)-fuzzy right ideals and \((\epsilon, \in \vee \mu_k)\)-fuzzy ideals are defined similarly.

**Example 0.13.** Consider \( S \) be as defined in Example 3.2. Define a fuzzy subset \( f \) of \( S \) by

\[
f(0) = 0.6, \quad f(a) = 0.6, \quad f(b) = 0.4 \quad \text{and} \quad f(c) = 0.2.
\]

Then \( f \) is an \((\epsilon, \in \vee \mu_k)\)-fuzzy ideal of \( S \).

**Theorem 0.14.** A fuzzy subset \( f \) of \( S \) is an \((\epsilon, \in \vee \mu_k)\)-fuzzy left (resp. right) ideal of \( S \) if and only if it satisfies the following conditions: for all \( x, y, z \in S \),

\[
(F3b) \max \left\{ f(x, y) \right\} \geq f(y), \quad (F4b) \text{If } x \leq y, \text{ then } \max \left\{ f(x) \right\} \geq f(y).
\]

**Proof.** We only consider the case for \((\epsilon, \in \vee \mu_k)\)-fuzzy left ideals, the case for \((\epsilon, \in \vee \mu_k)\)-fuzzy right ideals is similar.

Let \( f \) be an \((\epsilon, \in \vee \mu_k)\)-fuzzy left ideal of \( S \). If there exist \( x, y, z \in S \) such that \( f(x, y) \in \mu_k \) and \( f(y, z) \in \mu_k \), then \( f(xy) \in \mu_k \) and \( f(yz) \in \mu_k \). Hence \( (xy, z) \in f \). On the other hand, \((1 \circ f)(x, y) = \max \{ f(x), f(y) \} \geq f(y), \text{we have}(1 \circ f)(x, y) \geq f(y) \rightarrow f(y) = 1 - \frac{k}{2} \). Therefore \( F3b \) is satisfied. To prove \( F4b \) holds, let \( x \leq y \). If \( f(x) < f(y) \), then \( x \circ f \notin f \), \( y \circ f \notin f \), a contradiction. Thus \( F4b \) is valid.

Conversely, assume that the given conditions hold. If there exists \( x \circ f \) such that \( x \circ f \in \vee \mu_k \), then \( f(x) < f(y) \cdot (1 \circ f)(x) > f(y) \rightarrow f(y) = 1 - \frac{k}{2} \). Then \( f(x) < f(y) \cdot (1 \circ f)(x) > f(y) \rightarrow f(y) = 1 - \frac{k}{2} \). Hence \( f \) is an \((\epsilon, \in \vee \mu_k)\)-fuzzy left ideal of \( S \). □

**Proposition 0.16.** Let \( A \) be a non-empty subset of \( S \). Then \( A \) is a left (resp. right) ideal of \( S \) if and only if the fuzzy subset \( f \) of \( S \) such that \( f(x) = 1 \) for all \( x \in A \) and \( f(x) = \frac{1-k}{2} \) otherwise is an \((\epsilon, \in \vee \mu_k)\)-fuzzy left (resp. right) ideal of \( S \).

**Proof.** We only consider the case for \((\epsilon, \in \vee \mu_k)\)-fuzzy left ideals, the case for \((\epsilon, \in \vee \mu_k)\)-fuzzy right ideals is similar.

Assume that \( A \) is a left ideal of \( S \). Let \( x, y \in S \) and \( r \in [0, 1] \) be such that \( (xy) \in f \). Then \( f(xy) < r \leq 1 \) and so \( f(xy) = \frac{1-k}{2} < r \), i.e., \( xy \subseteq A \). It follows that \( y \subseteq A \) and so \( f(y) = \frac{1-k}{2} \). Hence \( y \subseteq f \), i.e., \( y \subseteq \vee \mu_k f \).

Now let \( x, y \in S \) and \( r \in [0, 1] \) be such that \( x < y \) and \( x \subseteq A \). Then \( f(xy) < r \leq 1 \) and so \( f(xy) = \frac{1-k}{2} < r \), i.e., \( x \subseteq A \). It follows that \( y \subseteq A \), and so \( f(y) = \frac{1-k}{2} \).

Hence \( y \subseteq f \), i.e., \( y \subseteq \vee \mu_k f \).
Therefore, \( f \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy left ideal of \( S \).

Conversely, assume that \( f \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy left ideal of \( S \). Let \( x \in S \) and \( y \in A \). Then \( f(y) = 1 \). It follows from \( \max\{f(xy), \frac{1}{1+y} \} \geq f(y) = 1 \) that \( f(xy) = 1 \), and so \( xy \in A \). Similarly, we may show that \( x \leq y \) and \( y \in A \) imply \( x \in A \). Therefore, \( A \) is a left ideal of \( S \). \( \square \)

**Proposition 0.17.** Let \( A \subseteq S \). Then \( A \) is a left (resp. right) ideal of \( S \) if and only if \( f_A \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy left (resp. right) ideal of \( S \).

**Proof.** The proof is analogous to that of Proposition 3.11. \( \square \)

**NEW FUZZY BI-IDEALS OF ORDERED SEMIGROUPS**

**Definition 0.18.** A fuzzy subset \( \mu \) of \( S \) is called an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal if it satisfies condition (F2a) and

\[
(F5a) \ f \circ f \subseteq \mathbb{Q}_k f.
\]

\[
(F6a) \ f \circ 1 \subseteq \mathbb{Q}_k f.
\]

**Example 0.19.** Consider \( S \) as defined in Example 3.2. Define a fuzzy subset \( f \) of \( S \) by

\[
f(0) = 0.4, \ f(a) = 0.2, \ f(b) = 0.6 \text{ and } f(c) = 0.2.
\]

Then \( f \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \), but it could neither be an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy left ideal of \( S \), nor an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy right ideal of \( S \).

The proof of the following results is similar to that in Section 3.

**Theorem 0.20.** A fuzzy subset \( f \) of \( S \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \) if and only if for all \( x, y, z \in S \), it satisfies condition (F2b) and

\[
F(5b) \ f(xy) \geq \min\{f(x), f(y), \frac{1}{1+y} \},
\]

\[
F(6b) \ f(xyz) \geq \min\{f(x), f(z), \frac{1}{1+z} \}.
\]

**Theorem 0.21.** A fuzzy subset \( f \) of \( S \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \) if and only if for all \( x, y, z \in S \) and \( r \in (0, 1] \), it satisfies (F2a) and

\[
F(5c) \ x_r \in f \text{ and } y_t \in f \text{ imply } (xy)_{\min(r,t)} \in \mathbb{Q}_k f,
\]

\[
F(6c) \ x_r \in f \text{ and } z_t \in f \text{ imply } (xyz)_{\min(r,t)} \in \mathbb{Q}_k f.
\]

**Proposition 0.22.** Let \( A \) be a non-empty subset of \( S \). Then \( A \) is a bi-ideal of \( S \) if and only if the fuzzy subset \( f \) of \( S \) such that \( f(x) = 1 \) for all \( x \in A \) and \( f(x) = \frac{1}{1+x} \) otherwise is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \).

**Proposition 0.23.** Let \( A \subseteq S \). Then \( A \) is a bi-ideal of \( S \) if and only if \( f_A \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \).

Next, we consider another generalized fuzzy bi-ideal of \( S \), which is called an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal.

**Definition 0.24.** A fuzzy subset \( f \) of \( S \) is called an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal if it satisfies condition (F4a) and

\[
(F7a) \ f \subseteq \mathbb{Q}_k f \circ f,
\]

\[
(F8a) \ f \subseteq \mathbb{Q}_k f \circ 1 \circ f.
\]

**Example 0.25.** Consider \( S \) as defined in Example 3.2. Define a fuzzy subset \( f \) of \( S \) by

\[
f(0) = 0.6, \ f(a) = 0.4, \ f(b) = 0.6 \text{ and } f(c) = 0.2.
\]

Then \( f \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \), but it could neither be an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy left ideal of \( S \), nor an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy right ideal of \( S \).

**Theorem 0.26.** A fuzzy subset \( f \) of \( S \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \) if and only if for all \( x, y, z \in S \), it satisfies condition (F4b) and

\[
(F7b) \ f(x, y) \geq \min\{f(x), f(y), \frac{1}{1+y} \},
\]

\[
(F8b) \ f(x, y, z) \geq \min\{f(x, y), f(z), \frac{1}{1+z} \}.
\]

**Theorem 0.27.** A fuzzy subset \( f \) of \( S \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \) if and only if for all \( x, y, z \in S \) and \( r \in (0, 1] \), it satisfies (F2a) and

\[
(F7c) \ (xy)_{\min(r,t)} \in f \text{ implies } x_r \in \mathbb{Q}_k f \text{ or } y_t \in \mathbb{Q}_k f.
\]

\[
(F8c) \ (xyz)_{\min(r,t)} \in f \text{ implies } x_r \in \mathbb{Q}_k f \text{ or } z_t \in \mathbb{Q}_k f.
\]

**Proposition 0.28.** Let \( A \) be a non-empty subset of \( S \). Then \( A \) is a bi-ideal of \( S \) if and only if the fuzzy subset \( f \) of \( S \) such that \( f(x) = 1 \) for all \( x \in A \) and \( f(x) = \frac{1}{1+x} \) otherwise is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \).

**Proposition 0.29.** Let \( A \subseteq S \). Then \( A \) is a bi-ideal of \( S \) if and only if \( f_A \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \).

From Theorems 3.3, 3.8, 4.3 and 4.8, we obtain the following result.

**Theorem 0.30.** (1) Every \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy right ideal of \( S \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \).

(2) Every \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy left (resp. right) ideal of \( S \) is an \( (\mathbb{T}, \mathbb{Z} \lor \mathbb{Q}) \)-fuzzy bi-ideal of \( S \).

Examples 4.2 and 4.8 indicate that the converse of Theorem 4.13(1) and 4.13(1) are not true in general.

**THE CHARACTERIZATION OF REGULAR ORDERED SEMIGROUPS IN TERMS OF NEW FUZZY IDEALS**

An ordered semigroup \( S \) is called regular if for every \( x \in S \) there exists \( a \in S \) such that \( x \leq xax \).

Equivalent definitions: (1) \( x \in (xSx) \forall x \in S \), (2) \( A \subseteq (ASA) \forall A \subseteq S \).

**Lemma 0.31.** [17] An ordered semigroup \( S \) is regular if and only if for every right ideal \( R \) and every left ideal \( L \) we have \( R \cap L \subseteq (RL) \).
Theorem 0.32. Let $S$ be an ordered semigroup. Then the following conditions are equivalent.

(1) $S$ is regular.

(2) $f \land g \subseteq \forall_{q} f \circ g$ for every $(\varepsilon, \in \forall_{q})$-fuzzy right ideal $f$ and every $(\varepsilon, \in \forall_{q})$-fuzzy left ideal $g$.

(3) $f \circ g \supseteq \forall_{\overline{q}} f \land g$ for every $(\varepsilon, \in \forall_{\overline{q}})$-fuzzy right ideal $f$ and every $(\varepsilon, \in \forall_{\overline{q}})$-fuzzy left ideal $g$.

Proof. (1)$(\Rightarrow)(2)$ Assume that (1) holds. Let $f$ and $g$ be any $(\varepsilon, \in \forall_{q})$-fuzzy right ideal and any $(\varepsilon, \in \forall_{q})$-fuzzy left ideal of $S$, respectively. If possible, let $f \land g \subseteq \forall_{q} f \circ g$. Then there exists $x_{r} \in f \land g$ such that $x_{r} \subseteq \forall_{q} f \circ g$, thus $(f \circ g(x)) < r$ and $(f \circ g(x) + r + k < 1$, and so $(f \circ g(x)) < r$ and $(f \circ g(x) < \frac{1 + k}{2}$. Since $S$ is regular, there exists $a \in S$ such that $x \leq x_{ax}$, then

$$\min \left\{ r, \frac{1 + k}{2} \right\} > (f \circ g(x)) = \bigvee_{(y,z) \in A_{s}} \min\{f(y), g(z)\} \geq \min\{f(x, r), g(x)\} \geq \min\left\{ \min\left\{ f(x), \frac{1 + k}{2} \right\} \right\} = \min\{r, \frac{1 + k}{2}\},$$

a contradiction. Therefore $f \land g \subseteq \forall_{q} f \circ g$.

(2)$(\Rightarrow)(1)$ Assume that (2) holds. Let $R$ and $L$ be any right ideal and any left ideal of $S$, respectively. Then Proposition 3.6 gives that $f_{R}$ and $f_{S}$ are an $(\varepsilon, \in \forall_{Q})$-fuzzy right ideal and an $(\varepsilon, \in \forall_{Q})$-fuzzy left ideal of $S$, respectively. By assumption and Lemma 2.5, we have $f_{R \land S} = f_{R} \land f_{S} \subseteq \forall_{q} f_{R} \circ f_{S} = f_{B_{1}}$, and so $R \cap L \subseteq (RL)$. Hence $S$ is regular by Lemma 5.1.

In a similar way, we may show that (1)$\Leftrightarrow$(3). □

Corollary 0.33. Let $S$ be an ordered semigroup. Then the following conditions are equivalent.

(1) $S$ is regular.

(2) $f \land g \subseteq \forall_{q} f \circ g$ for every fuzzy subset $f$ and every $(\varepsilon, \in \forall_{q})$-fuzzy left ideal $g$ of $S$.

(3) $f \land g \subseteq \forall_{q} f \circ g$ for every $(\varepsilon, \in \forall_{q})$-fuzzy right ideal $f$ and every fuzzy subset $g$ of $S$.

(4) $f \circ g \supseteq \forall_{\overline{q}} f \land g$ for every fuzzy subset $f$ and every $(\varepsilon, \in \forall_{\overline{q}})$-fuzzy left ideal $g$ of $S$.

(5) $f \circ g \supseteq \forall_{\overline{q}} f \land g$ for every $(\varepsilon, \in \forall_{\overline{q}})$-fuzzy right ideal $f$ and every fuzzy subset $g$ of $S$.

Lemma 0.34. An ordered semigroup $S$ is regular if and only if for every bi-ideal $B$ of $S$ we have $B \subseteq (BSB)$.

Proof. Assume that $S$ is regular. Let $B$ be any bi-ideal of $S$ and $x$ any element of $B$. Then there exists $a \in S$ such that $x \leq x_{ax}$. It is easy to see that $x_{ax} \in BSB$ and so $x \in (BSB)$. Hence $B \subseteq (BSB)$.

Conversely, assume that the given condition holds. Let $R$ and $L$ be any right ideal and any left ideal of $S$, respectively. Then it is easy to see that $R \cap L$ is a bi-ideal of $S$. By the assumption and Lemma 2.1, we have $R \cap L = ((R \cap L)S(R \cap L)) \subseteq (RL) \subseteq (RL)$. Hence $S$ is regular by Lemma 5.1. □

Theorem 0.35. Let $S$ be an ordered semigroup. Then the following conditions are equivalent.

(1) $S$ is regular.

(2) $f \subseteq \forall_{q} f \circ f$ for every $(\varepsilon, \in \forall_{q})$-fuzzy bi-ideal $f$ of $S$.

(3) $f \circ f \subseteq \forall_{\overline{q}} f \circ f$ for every $(\varepsilon, \in \forall_{\overline{q}})$-fuzzy bi-ideal $f$ of $S$.

Proof. (1)$(\Rightarrow)(2)$ Assume that (1) holds. Let $f$ be any $(\varepsilon, \in \forall_{q})$-fuzzy bi-ideal of $S$. If possible, let $f \subseteq \forall_{q} f \circ f$. Then there exists $x_{r} \in f$ such that $x_{r} \subseteq \forall_{q} f \circ f \circ f$, thus $(f \circ f)(x) < r$ and $(f \circ f)(x) < r + k < 1$, it follows that $(f \circ f)(x) < r$ and $(f \circ f)(x) < \frac{1 + k}{2}$. Since $S$ is regular, there exists $a \in S$ such that $x \leq x_{ax}$. Hence we have

$$r > (f \circ f)(x) = \bigvee_{(y,z) \in A_{s}} \min\{f(1)(y), f(z)\} \geq \min\{f(1)(xa), f(x)\} \geq \min\{f(x), f(x)\} = f(x) \geq r,$$

a contradiction. Hence $f \subseteq \forall_{q} f \circ f \circ f$.

(2)$(\Rightarrow)(1)$ Assume that (2) holds. Let $B$ be any bi-ideal of $S$. Then by Proposition 3.5, the characteristic function $f_{B}$ of $B$ is an $(\varepsilon, \in \forall_{q})$-fuzzy bi-ideal of $S$. Then, by the assumption and Lemma 2.1, we have $f_{B} \subseteq \forall_{q} f_{B} \circ f_{B} \circ f_{B} = f_{B_{BSB}}$, hence it follows from Lemma 2.5 that $B \subseteq (BSB)$. Hence $S$ is regular by Lemma 5.4.

In a similar way, we may show that (1)$\Leftrightarrow$(3). □

Theorem 0.36. Let $S$ be an ordered semigroup. Then the following conditions are equivalent.

(1) $S$ is regular.

(2) $f \land g \subseteq \forall_{q} f \circ g$ for every $(\varepsilon, \in \forall_{q})$-fuzzy bi-ideal $f$ and every $(\varepsilon, \in \forall_{q})$-fuzzy left ideal $g$ of $S$.

(3) $f \circ g \supseteq \forall_{\overline{q}} f \land g$ for every $(\varepsilon, \in \forall_{\overline{q}})$-fuzzy bi-ideal $f$ and every $(\varepsilon, \in \forall_{\overline{q}})$-fuzzy left ideal $g$ of $S$.

Proof. Assume that (1) holds. Let $f$ be any $(\varepsilon, \in \forall_{q})$-fuzzy bi-ideal of $S$. If possible, let $f \land g \subseteq \forall_{q} f \circ f$. Then there exists $x_{r} \in f \land g$ such that $x_{r} \subseteq \forall_{q} f \circ f \circ f$, thus $(f \circ f)(x) < r$ and $(f \circ f)(x) < r + k < 1$, it follows that $(f \circ f)(x) < r$ and $(f \circ f)(x) < \frac{1 + k}{2}$. Since $S$ is regular, there exists $a \in S$ such that $x \leq x_{ax}$.
Hence we have
\[
\min \left\{ r, \frac{1-k}{2} \right\} > (f \circ g \circ f)(x)
\]
\[
= \sum_{(y,z) \in A_x} \min\{f(y), g(z)\}
\geq \min\{f(x), g(ax)\}
\geq \min\left\{ f(x), \min \left\{ g(x), \frac{1-k}{2} \right\} \right\}
\geq \min\left\{ r, \frac{1-k}{2} \right\},
\]
a contradiction. Hence \( f \land g \subseteq V_{q_k} f \circ g \circ f. \)

(2)\(\Rightarrow\)(1) Assume that (2) holds. Let \( f \) be any \((\in, \in \in V_{q_k})\)-fuzzy bi-ideal of \( S. \) Then, since \( 1 \) is an \((\in, \in \in V_{q_k})\)-fuzzy ideal of \( S, \) we have \( f = f \land 1 \subseteq V_{q_k} f \circ 1 \circ f. \)
Thus it follows from Theorem 5.5 that \( S \) is regular.

In a similar way, we may show that (1)\(\Leftrightarrow\)(3). \( \square \)

**Theorem 0.37.** Let \( S \) be an ordered semigroup. Then the following conditions are equivalent.

1. \( S \) is regular.
2. \( f \land g \subseteq V_{q_k} f \circ g \) for every \((\in, \in \in V_{q_k})\)-fuzzy bi-ideal \( f \) and every \((\in, \in \in V_{q_k})\)-fuzzy left ideal \( g \) of \( S. \)
3. \( f \land g \subseteq V_{q_k} f \circ g \) for every \((\in, \in \in V_{q_k})\)-fuzzy right ideal \( f \) and every \((\in, \in \in V_{q_k})\)-fuzzy bi-ideal \( g \) of \( S. \)
4. \( f \land g \land h \subseteq V_{q_k} f \circ g \circ h \) for every \((\in, \in \in V_{q_k})\)-fuzzy right ideal \( f \), every \((\in, \in \in V_{q_k})\)-fuzzy bi-ideal \( g \) and every \((\in, \in \in V_{q_k})\)-fuzzy left ideal \( h \) of \( S. \)

**Proof.** (1)\(\Rightarrow\)(2) Assume that (1) holds. Let \( f \) and \( g \) be any \((\in, \in \in V_{q_k})\)-fuzzy left ideal and any \((\in, \in \in V_{q_k})\)-fuzzy bi-ideal of \( S, \) respectively. If possible, let \( f \land g \subseteq V_{q_k} f \circ g. \) Then there exists \( x \in f \land g \) such that \( x \in V_{q_k} f \circ g, \) thus \((f \circ g)(x) < r \) and \((f \circ g)(x) + r + k \leq 1), \) it follows that \((f \circ g)(x) < r \) and \((f \circ g)(x) < 1-k). \) Since \( S \) is regular, there exists \( a \in S \) such that \( x \leq xa. \) Hence we have
\[
\min \left\{ r, \frac{1-k}{2} \right\} > (f \circ g \circ f)(x)
\]
\[
= \sum_{(y,z) \in A_x} \min\{f(y), g(z)\}
\geq \min\{f(x), g(ax)\}
\geq \min\left\{ f(x), \min \left\{ g(x), \frac{1-k}{2} \right\} \right\}
\geq \min\left\{ r, \frac{1-k}{2} \right\},
\]
a contradiction. Hence \( f \land g \subseteq V_{q_k} f \circ g. \)

(2)\(\Rightarrow\)(1) Assume that (2) holds. Let \( f \) and \( g \) be any \((\in, \in \in V_{q_k})\)-fuzzy right ideal and any \((\in, \in \in V_{q_k})\)-fuzzy left ideal of \( S, \) respectively. Then, it is easy to see that \( f \) is an \((\in, \in \in V_{q_k})\)-fuzzy bi-ideal of \( S. \) By the assumption, we have \( f \land g \subseteq V_{q_k} f \circ g. \) Thus it follows from Theorem 5.2 that \( S \) is regular and (2) implies (1).

Similarly, we can show that (1)\(\Leftrightarrow\)(3).

(1)\(\Rightarrow\)(4) Assume that (1) holds. Let \( f, g \) and \( h \) be any \((\in, \in \in V_{q_k})\)-fuzzy right ideal, any \((\in, \in \in V_{q_k})\)-fuzzy bi-ideal and any \((\in, \in \in V_{q_k})\)-fuzzy left ideal of \( S, \) respectively. If possible, let \( f \land g \subseteq V_{q_k} f \circ g. \) Then there exists \( x \in f \land g \land h \) such that \( x \in V_{q_k} f \circ g \circ h, \) thus \((f \circ g \circ h)(x) < r \) and \((f \circ g \circ h)(x) + r + k \leq 1), \) it follows that \((f \circ g \circ h)(x) < r \) and \((f \circ g \circ h)(x) < 1-k). \) Since \( S \) is regular, there exists \( a \in S \) such that \( x \leq xa. \) Hence we have
\[
\min \left\{ r, \frac{1-k}{2} \right\} > (f \circ g \circ h)(x)
\]
\[
= \sum_{(y,z) \in A_x} \min\{f(y), g(z)\}
\geq \min\{f(x), g(ax)\}
\geq \min\left\{ f(x), \min \left\{ g(x), \frac{1-k}{2} \right\} \right\}
\geq \min\left\{ r, \frac{1-k}{2} \right\},
\]
a contradiction. Hence \( f \land g \land h \subseteq V_{q_k} f \circ g \circ h. \)

(4)\(\Rightarrow\)(1) Assume that (4) holds. Let \( f \) and \( g \) be any \((\in, \in \in V_{q_k})\)-fuzzy right ideal and any \((\in, \in \in V_{q_k})\)-fuzzy left ideal of \( S, \) respectively. Since \( 1 \) is an \((\in, \in \in V_{q_k})\)-fuzzy bi-ideal of \( S, \) by the assumption and Lemma 2.4, we have \( f \land g = f \land 1 \subseteq V_{q_k} f \circ 1 \circ g \subset V_{q_k} f \circ g. \) and so \( f \land g \subseteq V_{q_k} f \circ g \) by Lemma 2.3. Combining this with Theorem 5.2, we know that \( S \) is regular. \( \square \)

**Theorem 0.38.** Let \( S \) be an ordered semigroup. Then the following conditions are equivalent.

1. \( S \) is regular.
2. \( f \circ g \subseteq V_{q_k} f \land g \) for every \((\in, \in \in V_{q_k})\)-fuzzy bi-ideal \( f \) and every \((\in, \in \in V_{q_k})\)-fuzzy left ideal \( g \) of \( S. \)
3. \( f \circ g \subseteq V_{q_k} f \land g \) for every \((\in, \in \in V_{q_k})\)-fuzzy right ideal \( f \) and every \((\in, \in \in V_{q_k})\)-fuzzy bi-ideal \( g \) of \( S. \)
4. \( f \circ g \circ h \subseteq V_{q_k} f \circ g \circ h \) for every \((\in, \in \in V_{q_k})\)-fuzzy right ideal \( f \), every \((\in, \in \in V_{q_k})\)-fuzzy bi-ideal \( g \) and every \((\in, \in \in V_{q_k})\)-fuzzy left ideal \( h \) of \( S. \)

**Proof.** The proof is analogous to that of Theorem 5.7. \( \square \)

**Acknowledgements**
This paper was supported by the Natural Science Foundation for Young Scholars of Jiangxi, China (2010GQS0003).

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