Relativistic Problem by Choosing Spatially-Dependent Mass Coupled with a Tensor Potential

Mehdi Eshghi

Department of Physics, Central Tehran Branch, Islamic Azad University, Tehran, Iran

Abstract: In this research, we have solved the Dirac equation with spin symmetry for Pöschl-Teller potential including a Coulomb-like tensor interaction by choosing a spatially-dependent mass. The energy eigenvalues equation and the corresponding unnormalized wave functions have obtained in terms of the Jacobi polynomials. The Nikiforov-Uvarov method had used in the calculations. Some numerical results are given for this potential.

Key words: Dirac equation • Pöschl-Teller potential • Coulomb-like • Spatially-dependent mass • Nikiforov-Uvarov.

INTRODUCTION

Concepts of spin symmetry, pseudo-spin symmetry and a tensor potential have been found interesting applications in the field of nuclear physics [1-5]. Tensor potentials were introduced into the Dirac equation with the substitution \( \hat{p} \rightarrow \hat{p} - imab \beta \gamma U(r) \) [6, 7]. In this way, a spin-orbit coupling term is added to the Dirac Hamiltonian. Recently, tensor couplings have been used widely in the studies of nuclear properties. In this regard, see [8-16].

In recent years, the solution of Dirac, Klein-Gordon and Schrödinger equations with a spatially-dependent mass (SDM) are useful for the investigation of some physical systems. They are used, for example, in the determination of the electronic properties of the semiconductors [17], He clusters [18], in quantum liquids [19], in quantum dotes [20], etc. Some authers have investigated the exact solutions of the Dirac equation with position-depedent mass [12, 21-25].

According to the report which was given in the research [21, 26] the SDM for \( q = 1 \) of the form is

\[
M(r) = m_0 + 4V_0 \frac{e^{-2\alpha r}}{(1 + e^{-2\alpha r})^2}
\]

and the Pöschl-Teller potential of the form [26-28] is

\[
V(r) = -\frac{V_0}{\cosh^2 \alpha r}
\]

and tensor potential Coulomb-like [8] is

\[
U(r) = -\frac{H}{r}, \quad H = \frac{Z_aZ_b\epsilon^2}{4\pi\epsilon_0}, \quad r \geq R_c
\]

where \( R_c = 7.78 \text{ fm} \) is the Coulomb radius, \( Z_a \) and \( Z_b \) denote the charges of the projectile \( a \) and the target nuclei \( b \), respectively.

Our aim in this paper is study the Dirac equation for Pöschl-Teller potential including a Coulomb-like tensor coupling in the case of SDM distribution Eq. (1) under the spin symmetry. We have obtained the energy eigenvalues equation and the corresponding spinor wave functions by using the Nikiforov-Uvarov (NU) method.

Review of the Nikiforov-Uvarov Method: We give a brief description of the conventional NU method [29]. Recently, this method has been introduced for solving the Schrödinger, Klien-Gordon and Dirac equations with the well known potentials. For example, see [30-37].

The NU method reduces the second order differential equations to the hypergeometric type with an appropriate coordinate transformation \( s = s(r) \) as

\[
\frac{\psi_n''(s)}{\sigma(s)} + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0
\]

where \( \sigma(s) \) and \( \tilde{\sigma}(s) \) are polynomials, at the most of the second degree and \( \tilde{\tau}(s) \) is a polynomials, at most of the first degree. If we take the following factorization \( \psi_n(s) = \phi(s)y_n(s) \) (4) becomes

Corresponding Author: Mehdi Eshghi, Department of Physics, Central Tehran Branch, Islamic Azad University, Tehran, Iran. Tel.: +98-21-73026693, Fax: +98-21-77104938.
where
\[ \sigma(s)y_n(s) + \tau(s)y_n(s) + \lambda y_n(s) = 0 \] (5)
where \( \sigma(s) = \pi(s) \frac{d}{ds} (\ln \phi(s)) \) (6)
\[ \tau(s) = \tau(s) + 2\pi(s), \quad \tau'(s) < 0 \] (7)
where the Rodrigues formula
\[ \psi_n(s) = \frac{\alpha_n}{\rho(s)} \frac{d^n}{ds^n} \left[ \sigma^n(s) \rho(s) \right] \] (8)
where \( \alpha_n \) is a normalization constant and the weight function \( \rho(s) \) must satisfy the differential equation
\[ \omega'(s) = \left( \frac{\tau(s)}{\sigma(s)} \right) \omega(s) = 0, \omega(s) = \sigma(s) \rho(s). \] (9)

The function \( \pi(s) \) and the parameter \( \lambda \) in the above equation are defined as follows
\[ \pi(s) = \frac{\sigma'(s) - \tau(s)}{2} \pm \sqrt{\left( \frac{\sigma'(s) - \tau(s)}{2} \right)^2 - \sigma(s) + 4\pi(s)} \] (10)
and
\[ = q + \pi'(s) \] (11)
The determination of \( q \) is the essential point in the calculation of \( \pi(s) \). It is simply defined by setting the discriminant of the square root must be zero. The eigenvalues equation have calculated from the above equation
\[ \lambda = \lambda_m = -n\pi'(s) - \frac{n(n-1)}{2} \sigma'(s). \quad n = 0, 1, 2, \ldots \] (12)

**Dirac Equation:** According to the report which was given by researcher [8], the spatially-dependent mass Dirac equation including tensor interaction for spin-1/2 particles with both the scalar and the vector potential, in units where \( \hbar = c = 1 \), is
\[ [\hat{\alpha} \cdot \hat{P} + \beta (M(r) + V_s(r)) - i\beta \hat{\alpha} \cdot \hat{U}(r)] \psi_{nk}(r) = [E - V_s(r)] \psi_{nk}(r), \] (13)
where \( \alpha \) and \( \beta \) the 4x4 matrices, \( E \) is the relativistic energy of the system, \( \hat{P} = -\hat{\nabla} \) is the three-dimensional momentum operator. For a particle in a central field, the total momentum operator \( \hat{J} \) and operator \( \hat{K} \) is the spin-orbit matrix operator and have written in terms of the orbital angular momentum operator \( \hat{L} \) as \( \hat{K} = -\beta(\hat{\sigma} \cdot \hat{L} + 1) \), which commute with the Dirac Hamiltonian. For a given total angular momentum \( j \), the eigenvalues of \( \hat{K} \) are \( \kappa = -(j + 1/2) \) for aligned spin (\( s_{22}, p_{22}, \ldots \)) and \( \kappa = (j + 1/2) \) for unaligned spin (\( p_{22}, d_{32}, \ldots \)). By using the radial quantum number \( n \) and spin-orbit coupling quantum number \( k \), the Dirac spinors wave functions can be classified and had given by
\[ \psi_{nk}(r) = \frac{1}{r} \left[ F_{nk}(r) Y_{jm}(\theta, \phi) \right] \] (14)
where \( F_{nk}(r) \) is upper and \( G_{nk}(r) \) is the lower radial wave functions of the Dirac spinors, \( Y_{jm}(\theta, \phi) \) and \( Y_{jm}(\theta, \phi) \) are the spin and pseudo-spin spherical harmonics, respectively and \( m \) is the projection of the total angular momentum on the z-axis. The orbital angular-momentum quantum number \( l \) and \( j \) are the labels of upper and lower components. For a given spin-orbit momentum quantum number \( k = \pm 1, \pm 2, \pm 3, \ldots \), the orbital angular momentum and pseudo-orbital angular momentum are given by \( l = |k + 1/2| - 1/2 \) and \( j = |k - 1/2| - 1/2 \), respectively. Substituting (14) into (13) and using the following relations [38],
\[ (\hat{A} \cdot \hat{B}) = \hat{A} \cdot \hat{B} + i(\hat{A} \times \hat{B}) \] (15)
\[ (\hat{A} \cdot \hat{P}) = \hat{A} \cdot \hat{P} + i(\hat{A} \times \hat{r}) \] (16)
and properties
\[ (\hat{\sigma} \cdot \hat{L}) Y_{jm}(\theta, \phi) = (\kappa - 1) Y_{jm}(\theta, \phi) \] (17)
\[ (\hat{\sigma} \cdot \hat{L}) Y_{jm}(\theta, \phi) = (\kappa - 1) Y_{jm}(\theta, \phi) \] (18)
\[ (\hat{\sigma} \cdot \hat{L}) Y_{jm}(\theta, \phi) = -j Y_{jm}(\theta, \phi) \] (19)
\[ (\hat{\sigma} \cdot \hat{L}) Y_{jm}(\theta, \phi) = -j Y_{jm}(\theta, \phi) \] (20)
yields two coupled differential equations as follows
\[ \left( \frac{d}{dr} + \frac{K}{r} - U(r) \right) F_{nk}(r) = \left[ E_{nk} + M(r) - \Delta(r) \right] F_{nk}(r) \] (21)
\[ \left( \frac{d}{dr} - \frac{K}{r} + U(r) \right) G_{nk}(r) = \left[ M(r) - E_{nk} + \Sigma(r) \right] F_{nk}(r) \] (22)
where \( \Delta(r) = V_s(r) - V_s(r) \) and \( \Sigma(r) = V_s(r) + V_s(r) \). By substituting \( G_{nk}(r) \) from (21) into (22) and \( F_{nk}(r) \) from (22) into (21), we obtain the following two second-order differential equations for the upper and lower components,
In the above equations \( \kappa \kappa + 1 = l + 1 \) and \( \kappa \kappa - 1 = \hat{1}(l + 1) \). Equations (23) and (24) cannot be solved analytically because of the last term in the equations. It is convenient to solve the mathematical relation \([23, 39]\). By using this relation, we can exactly solve Eq. (23).

Substituting (1), (2) and (3) into (23) and considering spin symmetry and taking \( \Delta r = C \), as the Pöschl-Tellers potential and \( \Delta r = C \), i.e.\([40, 41]\), the equation obtained for the upper component of the Dirac spinor \( G_n(r) \) becomes

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa + 1}{r^2} + \frac{2 \kappa}{r} \frac{U(r)}{r} - \frac{dU(r)}{dr} - U^2(r) + (E_{nk} + M(r) - \Delta(r))(E_{nk} - M(r) - \Sigma(r)) \right\} G_{nk}(r) = 0.
\]

We take the following approximation \([42]\) as

\[
\frac{1}{r^2} \approx \frac{\alpha}{\sinh^2 \alpha r}
\]

By using a transformation of the form \( s = \tanh \alpha r \), we rewrite it as follows

\[
\left\{ \frac{d^2}{ds^2} + \frac{1 - 3s}{2s(1 - s)} \frac{d}{ds} + \frac{1}{[2s(1 - s)]^2} \right\} F_{nk}(s) = 0,
\]

where

\[
b_1 = (\kappa + H)(\kappa + H + 1) \quad \quad b_2 = \frac{E_{nk} + M_0 - C_s}{4\alpha^2} \quad \quad b_3 = M_0 - E_{nk}
\]

\[
\tilde{V}_0 = \frac{V_0}{4\alpha^2}
\]

By comparing (27) with (4), we determine polynomials as

\[
\sigma(s) = 2s(s - 1) \quad \quad \tilde{\sigma}(s) = 1 - 3s
\]

\[
\tilde{\sigma}(s) = -b_3\tilde{V}_0 s^2 + (b_1 - b_2b_3 - b_3\tilde{V}_0)s - b_1
\]

Substituting them into (10), we obtain

\[
\pi(s) = \frac{1 - s}{2}
\]

\[
\pm \frac{1}{2} \left\{ \left( -4\tilde{V}_0 b_3 - 8\tilde{\sigma} \right) s^2 + \left( 8q - 4(b_1 - b_3\tilde{V}_0 - b_2b_3) - 2 \right)s + 4b_1 + 1 \right\}
\]

The constant \( q \) is determined in the same way. Therefore, we get

\[
\pi(s) = \frac{1 - s}{2} \pm \frac{1}{2} \left\{ \left[ -2\sqrt{b_2b_3 + 1} + 4b_1 \right] s - \sqrt{1 + 4b_1} \right\}
\]

for

\[
q = \frac{-(b_1 + b_3\tilde{V}_0 + b_2b_3) + \sqrt{b_2b_3(1 + 4b_1)}}{2}
\]

and

\[
q = \frac{-(b_1 + b_3\tilde{V}_0 + b_2b_3) - \sqrt{b_2b_3(1 + 4b_1)}}{2}
\]

By using (31) and (12), we obtain the eigenvalue equation to be

\[
-\frac{1}{2} (b_1 + b_3\tilde{V}_0 + b_2b_3) - \frac{1}{2} \sqrt{b_2b_3(1 + 4b_1)} - \frac{1}{2} \left[ 2\sqrt{b_2b_3 + 1} + 4b_1 \right] - \frac{1}{2} = n \left[ 4 + 2\sqrt{b_2b_3 + 1} + 4b_1 \right] + 2n(n - 1)
\]

Some numerical results are given in table 1. We use the parameters \( C_s = 5, M = 1 \text{fm}^{-1}, \alpha = 0.01, V_0 = 10 \).

The wave function \( \phi(s) \) is obtained from (6) by taking \( \pi(s) \) and \( \sigma(s) \),

\[
\phi(s) = s^\frac{(k + H)}{2} \left( \frac{b_2b_3}{s} \right)^{\frac{k + H}{2}}
\]

and using (11), we have
Table 1: The bound state energy eigenvalues $E_{\pm}$ in unit of $\text{fm}^{-1}$ of the spin symmetry Pöschl-Teller potential for several values of $n$ and $k$. 

<table>
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<th>$n,k&lt;0$</th>
<th>$n,k&gt;0$</th>
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REFERENCES


