The Complete Weight Enumerator of LRTJ-Metric Array Codes over $\mathbb{Z}_4$

Sapna Jain$^1$ and K. P. Shum$^2$

1 Department of Mathematics, University of Delhi, Delhi 110 007, India
sapanjain@gmx.com

2 Institute of Mathematics, Yunnan University Kunming 650091, P.R. China
kpshum@ynu.edu.cn

Abstract: In [1], the first author introduced the notion of LRTJ-complete weight enumerator of array codes in the space $\text{Mat}_{m \times s}(\mathbb{Z}_q)$ ($q$ prime) which has already been introduced by the first author in [1]. In this paper, we first define the LRTJ-complete weight enumerator for quaternary array codes over the finite ring $\mathbb{Z}_4$ and then obtain the MacWilliams duality relation for the same.

Key words: Array codes, LRTJ-metric, MacWilliams identity

INTRODUCTION

The notion of LRTJ-complete weight enumerator of an array code in $\text{Mat}_{m \times s}(\mathbb{Z}_q)$ ($q$ prime) has already been introduced by the first author in [1]. In this paper, we first define the LRTJ-complete weight enumerator for quaternary array codes over the finite ring $\mathbb{Z}_4$ and then obtain the MacWilliams duality relation for the same.

We begin with the few definitions.

Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Let $\text{Mat}_{m \times s}(\mathbb{Z}_4)$ be the set of all $m \times s$ matrices with entries from $\mathbb{Z}_4$. Then $\text{Mat}_{m \times s}(\mathbb{Z}_4)$ is a module over $\mathbb{Z}_4$. Let $V$ be a $\mathbb{Z}_4$-submodule of the module $\text{Mat}_{m \times s}(\mathbb{Z}_4)$. Then $V$ is called an array code over $\mathbb{Z}_4$ (in fact, linear quaternary array code).

Definition 1. The Hamming weight $H(a)$ and Lee weight $L(a)$ of an element $a \in \mathbb{Z}_4$ is defined as

$$H(a) = \begin{cases} 1 & \text{if } a \neq 0, \\ 0 & \text{if } a = 0, \end{cases}$$

$$L(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a = 1, 3, \\ 2 & \text{if } a = 2. \end{cases}$$

We now define the LRTJ-metric in $\text{Mat}_{m \times s}(\mathbb{Z}_4)$ as follows:

Definition 2 [2,3]. Let $Y \in \text{Mat}_{1 \times s}(\mathbb{Z}_4)$ with $Y = (y_0, y_1, \cdots, y_{s-1})$. The LRTJ-weight of $Y$ denoted by

$$\tau(Y)$$

is defined as

$$\tau(Y) = \begin{cases} \max_{j=0,1,\cdots,s-1} \mathcal{L}(y_j) & \text{if } Y \neq 0, \\ 0 & \text{if } Y = 0, \end{cases}$$

where $\mathcal{L}(y_j) = \max_{j=0,1,\cdots,s-1} \{ j \ | \ y_j \neq 0 \}$.

Then $0 \leq \tau(Y) \leq s + 1$. Extending the definition of LRTJ-weight to the class of all $m \times s$ matrices as

$$\tau(A) = \sum_{i=1}^{m} \tau(A_i),$$

where $A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix} \in \text{Mat}_{m \times s}(\mathbb{Z}_4)$ and $A_i$ denotes the $i^{th}$ row of $A$. Then LRTJ-weight $\tau$ satisfies $0 \leq \tau(A) \leq m(s+1)$ for all $A \in \text{Mat}_{m \times s}(\mathbb{Z}_4)$ and determines a metric on $\text{Mat}_{m \times s}(\mathbb{Z}_4)$ if we set $d(A, A') = \tau(A - A')$ for all $A, A' \in \text{Mat}_{m \times s}(\mathbb{Z}_4)$ and is known as the LRTJ-metric (Lee-Rosenbloom-Tsfasman-Jain metric).

Definition 3[1]. Let $V \subseteq \text{Mat}_{m \times s}(\mathbb{Z}_4)$ be a linear quaternary array code. The LRTJ-weight spectrum of the code $V$ is the set

$$\{w_0, w_1, \cdots, w_{m(s+1)}\},$$

where for all $0 \leq r \leq m(s+1)$, $w_r$ is given by

$$w_r = |\{A \in V | \tau(A) = r\}|.$$
The LRTJ-weight enumerator of the code \( V \) is defined as

\[
W_V(z) = \sum_{r=0}^{m(s+1)} w_r z^r = \sum_{A \in V} z^{r(A)}.
\]

**Definition 4** [1]. Let \( Y_1 = (p_0, p_1, \ldots, p_{s-1}) \) and \( Y_2 = (q_0, q_1, \ldots, q_{s-1}) \) be two elements of \( \text{Mat}_{1 \times s}(Z_4) \). The inner product of \( Y_1 \) and \( Y_2 \) is defined by

\[
< Y_1, Y_2 > = \sum_{i=0}^{s-1} p_i q_{s-1-i},
\]

and this is extended to the inner product of \( A = (A_1, A_2, \ldots, A_m)^T \) and \( B = (B_1, \ldots, B_m)^T \in \text{Mat}_{m \times s}(Z_4) \) as

\[
< A, B >= \sum_{i=1}^{m} < A_i, B_i >.
\]

The dual of a linear quaternary array code \( V \subseteq \text{Mat}_{m \times s}(Z_4) \) is defined as

\[
V^\perp = \{ B \in \text{Mat}_{m \times s}(Z_4) \mid < A, B > = 0 \text{ for all } A \in V \}.
\]

Then \( V^\perp \subseteq \text{Mat}_{m \times s}(Z_4) \) is also a linear quaternary array code.

**Example 5.** Let \( V_1 \) and \( V_2 \) be two linear quaternary array codes in \( \text{Mat}_{1 \times 2}(Z_4) \) given by

\[
V_1 = \{(0,0), (0,2)\},
\]

\[
V_2 = \{(0,0), (2,2)\}.
\]

The LRTJ-weight enumerators of both \( V_1 \) and \( V_2 \) is \( 1+z^3 \).

The dual codes of \( V_1 \) and \( V_2 \) are given by

\[
V_1^\perp = \{(0,0), (0,1), (0,2), (0,3), (2,0), (2,1), (2,2), (2,3)\},
\]

\[
V_2^\perp = \{(0,0), (1,1), (2,2), (3,3), (0,2), (1,3), (2,0), (3,1)\}.
\]

The LRTJ-weight enumerators of \( V_1^\perp \) and \( V_2^\perp \) are

\[
W_{V_1^\perp}(z) = 1 + 3z^2 + 4z^3,
\]

\[
W_{V_2^\perp}(z) = 1 + 5z^2 + 2z^3.
\]

Thus we observe that although the LRTJ-weight enumerators of the codes \( V_1 \) and \( V_2 \) are the same, the LRTJ-weight enumerators of their duals are different. Motivated by this, we introduce the notion of LRTJ-complete weight enumerator for quaternary array codes:

**Definition 6.** Let \( V \subseteq \text{Mat}_{m \times s}(Z_4) \) be an \( m \times s \) quaternary array code over \( Z_4 \) having \( n \) code arrays. Let

\[
V = \{ A^{(1)}, A^{(2)}, \ldots, A^{(n)} \}.
\]

Also, for \( i \leq l \leq n \), let

\[
A^{(i)} = \begin{pmatrix}
   a_{10}^{(i)} & a_{11}^{(i)} & \cdots & a_{1,s-1}^{(i)} \\
   a_{20}^{(i)} & a_{21}^{(i)} & \cdots & a_{2,s-1}^{(i)} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{m0}^{(i)} & a_{m1}^{(i)} & \cdots & a_{m,s-1}^{(i)}
\end{pmatrix}.
\]

Let \( Y_{ms} \) and \( T_{ms} \) be two \( ms \)-tuples of variables given by

\[
Y_{ms} = (y_{10}, \ldots, y_{1,s-1}, \ldots, y_{m0}, \ldots, y_{ms})
\]

and

\[
T_{ms} = (t_{10}, \ldots, t_{1,s-1}, \ldots, t_{m0}, \ldots, t_{ms}).
\]

We define the LRTJ-complete weight enumerator of the \( m \times s \) quaternary array code \( V \) by

\[
W_V(Y_{ms}, T_{ms}) = \sum_{i=1}^{n} y_{10}^{H(a_{10}^{(i)})} t_{10}^{H(a_{11}^{(i)})} \cdots y_{1,s-1}^{H(a_{1,s-1}^{(i)})} t_{1,s-1}^{H(a_{2,s-1}^{(i)})} \cdots y_{m0}^{H(a_{m0}^{(i)})} t_{m0}^{H(a_{m1}^{(i)})} \cdots y_{ms}^{H(a_{m,s-1}^{(i)})} t_{ms}^{H(a_{m,s-1}^{(i)})}.
\]

Then the LRTJ-complete weight enumerator \( W_V(Y_{ms}, T_{ms}) \) of the quaternary array code \( V \) is a polynomial in \( 2ms \) variables. Further, it is possible to obtain the LRTJ-weight enumerator as a special case of the LRTJ-complete weight enumerator as shown in the following example:

**Example 7.** The LRTJ-complete weight enumerators of the quaternary array codes \( V_1, V_2, V_1^\perp \) and \( V_2^\perp \) of Example 1.1 are given by

\[
W_{V_1}(Y_{12}, T_{12}) = 1 + y_{11} t_{11}^2,
\]

\[
W_{V_2}(Y_{12}, T_{12}) = 1 + y_{10} t_{10} y_{11} t_{11}^2,
\]

\[
W_{V_1^\perp}(Y_{12}, T_{12}) = 1 + 2y_{11} t_{11}
\]

\[
+ y_{11} t_{11}^2 + y_{10} t_{10}^2 + 2y_{10} t_{10} y_{11} t_{11} + y_{10} t_{10}^2 y_{11} t_{11}^2 + y_{10}^2 t_{10} y_{11} t_{11}^2 + y_{10}^2 t_{10}^2 y_{11} t_{11}^2 + y_{10}^2 t_{10}^2 y_{11} t_{11}^2 + y_{10}^2 t_{10}^2 y_{11} t_{11}^2.
\]

\[
W_{V_2^\perp}(Y_{12}, T_{12}) = 1 + 4y_{10} t_{10} y_{11} t_{11}
\]

\[
+ y_{10}^2 t_{10} y_{11} t_{11}^2 + y_{10}^2 t_{10}^2 y_{11} t_{11}^2 + y_{10}^2 t_{10}^2 y_{11} t_{11}^2 + y_{10}^2 t_{10}^2 y_{11} t_{11}^2.
\]

By letting for each \( 1 \leq j \leq 2 \),

\[
y_{j0}^{i0} y_{j1}^{i1} y_{j2}^{i2} y_{j3}^{i3} = \begin{cases}
x^{2i_2+(1-i_3) i_0-1} & \text{if either of } i_0, i_2 \neq 0, \\
1 & \text{if } i_0 = i_2 = 0,
\end{cases}
\]
in the complete weight enumerators of codes $V_1, V_2, V_1^\perp$ and $V_2^\perp$, we obtain the LRTJ-weight enumerators of these codes discussed in Example 5.

In order to state and prove the MacWilliams type identity for the LRTJ-complete weight enumerator of a quaternary array code $V \subseteq Mat_{m \times s}(\mathbb{Z}_4)$, we make the following identification:

**Definition 8.** Define

$$\theta_1 : Mat_{1 \times s}(\mathbb{Z}_4) \rightarrow \mathbb{Z}_4[x]_{<x^s>}$$

as

$$\theta_1((p_0, p_1, \ldots, p_{s-1})) = p_0 + p_1 x + \cdots + p_{s-1} x^{s-1}.$$ 

Let $P = (P_1, P_2, \ldots, P_m)^T \in Mat_{m \times s}(\mathbb{Z}_4)$ where $P_i = (p_{i0}, p_{i1}, \ldots, p_{is-1})$ for all $i \leq m$.

Extending $\theta_1$ to $Mat_{m \times s}(\mathbb{Z}_4)$ as

$$\theta : Mat_{m \times s}(\mathbb{Z}_4) \rightarrow Mat_{m \times 1}(\mathbb{Z}_4[x]_{<x^s>})$$
given by

$$\theta(P) = (p_{00} + p_{01} x + \cdots + p_{0,s-1} x^{s-1}, \ldots, p_{m0} + p_{m1} x + \cdots + p_{m,s-1} x^{s-1})^T.$$ 

Then $\theta$ is a $\mathbb{Z}_4$-module space isomorphism.

The LRTJ-weight of a polynomial $p(x) \in \mathbb{Z}_4[x]_{<x^s>}$ is given by

$$\tau(p(x)) = \deg(p(x)) + \sum_{j=0}^{s-1} \mu_j(L(p_j)),$$

where $p(x) = p_0 + p_1 x + \cdots + p_{s-1} x^{s-1}$.

Further, we define the $l^{th}$ ($0 \leq l \leq s - 1$) coefficient of $p(x)$ as

$$a_l(p(x)) = p_l.$$

Let $P(x) = (P_1(x), \ldots, P_m(x))^T$ and $Q(x) = (Q_1(x), \ldots, Q_m(x))^T$ be two elements in $Mat_{m \times 1}(\mathbb{Z}_4[x]_{<x^s>})$ where for all $1 \leq i \leq m$,

$$P_i(x) = p_{i0} + p_{i1} x + \cdots + p_{is-1} x^{s-1}$$

and

$$Q_i(x) = q_{i0} + q_{i1} x + \cdots + q_{is-1} x^{s-1}.$$ 

The inner product of $P(x)$ and $Q(x)$ given in Definition 4 becomes

$$< P(x), Q(x) > = \sum_{i=1}^{m} c_{i-1}(P_i(x)Q_i(x)).$$

Now we define the character of the additive abelian group $(\mathbb{Z}_4, +)$.

**Definition 9** [4]. Let $U$ be the multiplicative group of complex numbers having absolute value 1 i.e.

$$U = \{ z \in \mathbb{C} : |z| = 1 \}.$$ 

Define $\chi : \mathbb{Z}_4 \rightarrow U$ by

$$\chi(\alpha) = i^\alpha \text{ } \forall \text{ } \alpha \in \mathbb{Z}_4$$

where $i = \sqrt{-1}$.

Then $\chi$ is a group homomorphism and hence a character of $\mathbb{Z}_4$.

**Observations**

1. $\sum_{\alpha \in \mathbb{Z}_4} \chi(\alpha) = 0$.
2. $\chi(0) + \chi(2) = 0$.

### 2. The LRTJ-Complete Weight Enumerator of the Dual Code of a Quaternary Array Code

In this section, we obtain the MacWilliams type identity for quaternary array codes i.e. given the LRTJ-complete weight enumerator of a quaternary array code $V$, we obtain the LRTJ-complete weight enumerator of its dual code $V^\perp$. To prove the desired identity, we first prove few lemmas. Throughout this section, the character $\chi$ of $\mathbb{Z}_4$ is taken according to Definition 9.

**Lemma 10.** Let $\chi$ be the character of $\mathbb{Z}_4$. Let $V \subseteq Mat_{m \times 1}(\mathbb{Z}_4[x]_{<x^s>})$ be a quaternary array code and $Q(x)$ be an element of $Mat_{m \times 1}(\mathbb{Z}_4[x]_{<x^s>})$. Then

$$\sum_{P(x) \in V} \chi(< P(x), Q(x) >)$$

$$= \left\{ \begin{array}{ll} 0 & \text{if } Q(x) \not\in V^\perp \\ |V| & \text{if } Q(x) \in V^\perp. \end{array} \right.$$ 

**Proof.** If $Q(x) \in V^\perp$, then clearly $< P(x), Q(x) >= 0$. This implies that

$$\sum_{P(x) \in V} \chi(< P(x), Q(x) >) = \sum_{P(x) \in V} \chi(0) = |V|.$$ 

If $Q(x) \not\in V^\perp$, then in the summation $\sum_{P(x) \in V} \chi(< P(x), Q(x) >)$, the inner product $< P(x), Q(x) >$ either assumes values 0, 1, 2, 3 equally often or values 0 and 2 equally often. In either case, we have

$$\sum_{P(x) \in V} \chi(< P(x), Q(x) >) = 0.$$
using observations after Definition 9). Q.E.D.

**Lemma 11.** Let $\chi$ be the character of $\mathbb{Z}_4$. Let $\beta$ be a fixed element of $\mathbb{Z}_4$. Then

$$
\sum_{\alpha \in \mathbb{Z}_4} \chi(\beta \alpha) y^{H(\alpha)L(\alpha)} =
\begin{cases}
1 + 2t y + y^2 & \text{if } \beta = 0, \\
1 - y^2 & \text{if } \beta = 1, 3, \\
1 - 2t y + y^2 & \text{if } \beta = 2.
\end{cases}
$$

**Proof.** Consider

$$
\sum_{\alpha \in \mathbb{Z}_4} \chi(\beta \alpha) y^{H(\alpha)L(\alpha)} = \chi(0) + \chi(\beta, 1) y^1 + \chi(\beta, 2) y^2 + \chi(\beta, 3) y^3
= \chi(0) + (\chi(\beta, 1) + \chi(\beta, 3)) y^1 + \chi(\beta, 2) y^2
= \begin{cases}
1 + 2t y + y^2 & \text{if } \beta = 0, \\
1 - y^2 & \text{if } \beta = 1, 3, \\
1 - 2t y + y^2 & \text{if } \beta = 2.
\end{cases}
$$

Q.E.D.

**Lemma 12.** Let $\chi$ be the character of $\mathbb{Z}_4$ and $i, j$ be fixed nonnegative integers. Let $p(x) = p_{00} + p_{10} x + \cdots + p_{i, s-1} x^{s-1} \in \mathbb{Z}_4[x] < x^s >$. Then

$$
\sum_{\alpha \in \mathbb{Z}_4} \chi(< p(x), \alpha x^j >) y_{ij}^{H(\alpha)L(\alpha)} =
\begin{cases}
1 + 2y_{ij} t_{ij} + y_{ij} t_{ij}^2 & \text{if } p_{i, s-1-j} = 0, \\
1 - y_{ij} t_{ij} & \text{if } p_{i, s-1-j} = 1, 3, \\
1 - 2y_{ij} t_{ij} + y_{ij} t_{ij}^2 & \text{if } p_{i, s-1-j} = 2.
\end{cases}
$$

**Proof.** Since

$$
\sum_{\alpha \in \mathbb{Z}_4} \chi(< p(x), \alpha x^j >) y_{ij}^{H(\alpha)L(\alpha)} =
\sum_{\alpha \in \mathbb{Z}_4} \chi(p_{s-1}(p(x) \times x^j)) y_{ij}^{H(\alpha)L(\alpha)} =
\sum_{\alpha \in \mathbb{Z}_4} \chi(p_{i, s-1-j} \times x) y_{ij}^{H(\alpha)L(\alpha)},
$$

the proof now follows from Lemma 11. Q.E.D.

**Theorem 14.** Let $V \subseteq Mat_{m \times 1} \left( \mathbb{Z}_4[x] < x^s > \right)$ be a quaternary array code equipped with the LRTJ-metric. Let $Q(x), P(x) \in Mat_{m \times 1} \left( \mathbb{Z}_4[x] < x^s > \right)$ be given as $Q(x) = (Q_1(x), \ldots, Q_m(x))^T$ with $Q_i(x) = q_{00} + q_{10} x + \cdots + q_{i, s-1} x^{s-1}$ and $P(x) = (P_1(x), \ldots, P_m(x))^T$ with $P_i(x) = p_{00} + p_{10} x + \cdots + p_{i, s-1} x^{s-1} (1 \leq i \leq m)$. Then

$$
\sum_{Q(x) \in V^\perp} y_{10}^{H(q_{10})L(q_{10})} \cdots
y_{1, s-1}^{H(q_{1, s-1})L(q_{1, s-1})} \cdots
y_{m0}^{H(q_{m0})L(q_{m0})} \cdots
y_{m, s-1}^{H(q_{m, s-1})L(q_{m, s-1})}
$$

where $\hat{f}$ is the Hadamard transform of $f$ given by

$$
\hat{f}(P(x)) =
\sum_{Q(x) \in Mat_{m \times 1} \left( \mathbb{Z}_4[x] < x^s > \right)} \chi(< P(x), Q(x) >) f(Q(x)),$n

and

$$
P(x) = (P_1(x), \ldots, P_m(x))^T, \
Q(x) = (Q_1(x), \ldots, Q_m(x))^T.
$$

**Proof.**
\[ \frac{1}{|V|} \sum_{P(x) \in V} \prod_{i=1}^{m} s^{-1} \]
\[ (1 + 2y_{ij}t_{ij} + y_{ij}t_{ij}^2)^{\delta_{E_{j_{ij}}}} \]
\[ \times \left(1 - y_{ij}t_{ij}^2\right)^{\delta_{E_{j_{ij}}}} \]
\[ \times \left(1 - 2y_{ij}t_{ij} + y_{ij}t_{ij}^2\right)^{\delta_{E_{j_{ij}}}} \], \]
\[ \text{where } \delta \text{ is the Kronecker delta.} \]

**Proof.** Take \( f : Mat_{m \times 1}(Z_4[x]_{< x^s >}) \rightarrow C[y_{10}, \ldots, y_{m, s-1}, t_{10}, \ldots, t_{m, s-1}] \) in Lemma 13 as

\[ f(Q(x)) = \sum_{Q(x) \in Y^s} f(Q(x)) \]
\[ = \frac{1}{|V|} \sum_{P(x) \in V} \hat{f}(P(x)). \]

(2) Applying Lemma 12, we get

\[ \hat{f}(P(x)) = \prod_{l=0}^{s-1} \left(1 + 2y_{l1}t_{l1} + y_{l1}t_{l1}^2\right)^{\delta_{E_{j_{l1}}}} \]
\[ \times \left(1 - y_{l1}t_{l1}^2\right)^{\delta_{E_{j_{l1}}}} \]
\[ \times \left(1 - 2y_{l1}t_{l1} + y_{l1}t_{l1}^2\right)^{\delta_{E_{j_{l1}}}} \]
\[ = \prod_{l=0}^{s-1} \sum_{Q(x) \in Mat_{m \times 1}(Z_4[x]_{< x^s >})} \prod_{i=1}^{m} s^{-1} \]
\[ \chi(< P_l(x), Q_l(x) >) \]
\[ \times \left(1 + 2y_{l1}t_{l1} + y_{l1}t_{l1}^2\right)^{\delta_{E_{j_{l1}}}} \]
\[ \times \left(1 - y_{l1}t_{l1}^2\right)^{\delta_{E_{j_{l1}}}} \]
\[ \times \left(1 - 2y_{l1}t_{l1} + y_{l1}t_{l1}^2\right)^{\delta_{E_{j_{l1}}}} \]
\[ = \prod_{l=0}^{s-1} \pi_{l=0}^{s-1} \]
\[ \times \left(1 + 2y_{ml}t_{ml} + y_{ml}t_{ml}^2\right)^{\delta_{E_{j_{ml}}}} \]
\[ \times \left(1 - y_{ml}t_{ml}^2\right)^{\delta_{E_{j_{ml}}}} \]
\[ \times \left(1 - 2y_{ml}t_{ml} + y_{ml}t_{ml}^2\right)^{\delta_{E_{j_{ml}}}} \]
\[ = \prod_{l=0}^{s-1} \pi_{l=0}^{s-1} \]
\[ = 1892 \]
\[\left(1 + 2y_{ij}t_{ij} + y_{ij}^2 t_{ij}^2\right) s^{3}_{i_1(l_{i_1}, s-1-j)} \times \\
\left(1 - y_{ij}^2 t_{ij}^2\right) s^{4}_{i_2(l_{i_2}, s-1-j)} \times \\
\left(1 - 2y_{ij}t_{ij} + y_{ij}^2 t_{ij}^2\right) s^{4}_{i_3(l_{i_3}, s-1-j)}.\]

Thus
\[
\sum_{P(x) \in V} = \sum_{P(x) \in V_i} \prod_{i=1}^{m} \sum_{j=0}^{s-1} \left(1 + 2y_{ij}t_{ij} + y_{ij}^2 t_{ij}^2\right) s^{4}_{i_1(l_{i_1}, s-1-j)} \times \\
\left(1 - y_{ij}^2 t_{ij}^2\right) s^{4}_{i_2(l_{i_2}, s-1-j)} \times \\
\left(1 - 2y_{ij}t_{ij} + y_{ij}^2 t_{ij}^2\right) s^{4}_{i_3(l_{i_3}, s-1-j)}.\]

From (2) and (3) we get (1). Q.E.D.

**Example 15.** Consider the codes \(V_1\) and \(V_2\) of Example 1.1. Here \(m = 1, s = 2, |V_1| = |V_2| = 2.\) We find \(\text{LRTJ-complete weight enumerator of the dual codes } V_1^+ \text{ and } V_2^+ \text{ using (1).}\)

\[
W_{V_1^+}(Y_{12}, T_{12}) = \frac{1}{2} \left[ (1 + 2y_{10}t_{10} + y_{10}t_{10}^2)(1 + 2y_{11}t_{11} + y_{11}t_{11}^2) + \\
(1 - 2y_{10}t_{10} + y_{10}t_{10}^2)(1 + 2y_{11}t_{11} + y_{11}t_{11}^2)\right]
\]

\[
= 1 + 2y_{11}t_{11} + y_{11}t_{11}^2 + \\
y_{10}t_{10} + 2y_{10}t_{10}y_{11}t_{11} + \\
y_{10}t_{10}y_{11}t_{11}.
\]

\[
W_{V_2^+}(Y_{12}, T_{12}) = \frac{1}{2} \left[ (1 + 2y_{10}t_{10} + y_{10}t_{10}^2)(1 + 2y_{11}t_{11} + y_{11}t_{11}^2) + \\
(1 - 2y_{10}t_{10} + y_{10}t_{10}^2)(1 - 2y_{11}t_{11} + y_{11}t_{11}^2)\right]
\]

\[
= 1 + y_{11}t_{11} + 4y_{10}t_{10}y_{11}t_{11} + \\
y_{10}t_{10} + y_{10}t_{10}y_{11}t_{11}^2,
\]

which coincides with the LRTJ-complete weight enumerator of \(V_1^+\) and \(V_2^+\) obtained in Example 7.

**Acknowledgment:** The first author would like to thank her husband Dr. Arifiant Jain for his constant support and encouragement for pursuing research.

**REFERENCES**


