Differential Transform Method for Abel Differential Equation

M. Esfami, H. Zareamoghaddam

Kashmar Branch, Islamic Azad University, Kashmar, Iran

Abstract: In this paper, an analytic solution is presented using differential transform method (DTM) for a class of nonlinear ODE. The emphasis is on the Abel differential equation. The procedures introduced in this paper are in recursive forms which can be used to obtain the closed form of the solutions, if they are required. The method is tested on various examples and the results reveal the effectiveness and simplicity of the method.

Key words: Differential transform method · Abel differential equation

INTRODUCTION

The concept of the differential transform was first proposed by Zhou [1] and its main applications therein are for solving both linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of a polynomial. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method computationally takes long time for high-order derivatives. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. In recent years, researchers have applied the method to various linear and nonlinear problems. For example, it was applied to partial differential equations [2], integro-differential equations [3], two point boundary value problems [4], differential-algebraic equations [5], the KdV and mKdV equations [6], the Schrödinger equations [7], the boundary value problems [8] and fractional differential equations [9].

In this paper, we propose differential transform method to solve Abel differential equation. Consider the following form of Abel differential equation

\[
\frac{dy}{dt} = \sum_{i=0}^{m} f_i(t) y^i.
\]  

This differential equation arose in the context of the studies of N.H. Abel [9], on the theory of elliptic functions. An interesting area of this type of equations can also be seen in biological systems, see [10] and in many physical problems, engineering, ecology and economics see [11]. Thus, methods of solution for these equations are of great importance to engineers and scientists. Abel differential equation represents a natural generalization of the Riccati equation.

Basic Idea of Differential Transform Method: The basic definitions and fundamental operations of differential transform are given in [1-8]. For convenience, we will present a review of the method. The differential transform function of \( K \phi \) derivative of a function \( u(x) \) is defined as follows

\[
\mathcal{U}(k) = \frac{1}{k!} \left[ \frac{d^k u(x)}{dx^k} \right]_{x=x_0},
\]

where \( u(x) \) is the original function and \( U(k) \) is the transformed function. The differential inverse transform of \( U(k) \) is defined as the following

\[
u(x) = \sum_{k=0}^{\infty} U(k)(x-x_0)^k.
\]

In a real application, when \( x_0 \) is taken as 0, then the function \( u(x) \) is expressed by a finite series and Eq. (3) can be written as follows

\[
u(x) = \sum_{k=0}^{\infty} U(k)x^k.
\]

The fundamental mathematical operations performed by one-dimensional differential transform method can be obtained readily and are listed in Table 1.

Corresponding Author: H. Zareamoghaddam, Kashmar Branch, Islamic Azad University, Kashmar, Iran. E-mail: hossein.zareamoghaddam@yahoo.com
Table 1:

<table>
<thead>
<tr>
<th>Original function</th>
<th>Transformed function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(x) = f(x) \pm g(x)$</td>
<td>$U(k) = G(k) \pm H(k)$</td>
</tr>
<tr>
<td>$u(x) = \lambda g(x)$</td>
<td>$U(k) = \lambda G(k)$</td>
</tr>
<tr>
<td>$u(x) = \frac{\partial g(x)}{\partial x}$</td>
<td>$U(k) = (k + 1)G(k + 1)$</td>
</tr>
<tr>
<td>$u(x) = \frac{\partial^n g(x)}{\partial x^n}$</td>
<td>$U(k) = (k + 1)(k + 2)\cdots(k + m)G(k + m)$</td>
</tr>
<tr>
<td>$u(x) = x^n$</td>
<td>$U(k) = \delta(k - m) = \begin{cases} 1, &amp; k = m \ 0, &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>$u(x) = f(x)g(x)$</td>
<td>$U(k) = \sum_{\tau=0}^{k} F(\tau)G(k - \tau)$</td>
</tr>
<tr>
<td>$u(x) = f_1(x)f_2(x)\cdots f_n(x)$</td>
<td>$U(k) = \sum_{k_1=0}^{k} \cdots \sum_{k_n=0}^{k} F_1(k_1)F_2(k_2 - k_1)\cdots F_n(k - k_{n-1})$</td>
</tr>
</tbody>
</table>

**Numerical Examples:** To illustrate the ability and reliability of the method for Abel differential equation, some examples are provided. The results reveal that the method is very effective and simple.

Example 1. Consider the following Abel differential equation

$$\frac{dy}{dt} = 1 + y^2(t), \quad (5)$$

subject to the initial condition

$$y(0) = 0. \quad (6)$$

With the exact solution $y(t) = \tan t$.

The Taylor expansion of $y(t)$, about $t = 0$ can be written as follows

$$y(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \cdots$$

Taking the differential transform of Eq. (5), leads to

$$(k + 1)Y(k + 1) = \delta(k) + \sum_{\tau=0}^{k} Y(\tau)Y(k - \tau). \quad (7)$$

From the initial condition, given by Eq. (6)

$$Y(0) = 0, \quad (8)$$

is obtained. Substituting Eq. (8) into Eq. (7) and by recursive method, the results are listed as follows

$$Y(1) = 1, \quad Y(2) = 0, \quad Y(3) = \frac{1}{3}, \quad Y(4) = 0, \quad Y(5) = \frac{2}{15}, \quad Y(6) = \frac{17}{315}, \quad \vdots$$

Substituting all $Y(k)$ into Eq. (4), we obtained the series solution as the following

$$y(t) = \sum_{k=1}^{\infty} Y(k)t^k = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \cdots = \tan t,$$

which is the exact solution of the equation.

Example 2. Consider the Abel equation

$$\frac{dy}{dt} = -y(t) + y^2(t), \quad (9)$$

subject to the initial condition

$$y(0) = \frac{1}{2} \quad (10)$$

With the exact solution $y(t) = \frac{e^{-t}}{1 + e^{-t}}$.

The Taylor expansion of $y(t)$ about $t = 0$ gives

$$y(t) = \frac{1}{2} - \frac{1}{4}t + \frac{1}{48}t^3 - \frac{1}{480}t^5 + \frac{17}{80640}t^7 - \frac{31}{1451200}t^9 + \cdots$$

1013
Taking the differential transform of Eq. (9), then

\[(k + 1)Y(k + 1) = -Y(k) + \sum_{r=0}^{k} Y(r)Y(k-r). \quad (11)\]

From Eq. (10),

\[y(0) = \frac{1}{2} \quad (12)\]

substituting Eq. (12) into Eq. (11) and by recursive method, the results are listed follows

\[
\begin{align*}
Y(1) &= -\frac{1}{4}, \\
Y(2) &= 0, \\
Y(3) &= \frac{1}{48}, \\
Y(4) &= 0, \\
Y(5) &= -\frac{1}{480}, \\
Y(6) &= 0, \\
&\vdots
\end{align*}
\]

Substituting all \(Y(k)\) into Eq. (4), we obtain the series solution form as follows

\[y(t) = \sum_{k=0}^{\infty} Y(k)t^k = \frac{1}{2} - \frac{1}{4}t + \frac{1}{48}t^2 - \frac{17}{80640}t^3 + \frac{31}{14120}t^4 + \cdots \]

which is the expansion of the function \(\frac{e^{-t}}{1+e^{-t}}\) and is the exact solution of the equation given in Eqs. (9)-(10).

Example 3. Consider the following equation

\[
\frac{dy}{dt} = -2y - t^4 - y(t) + t^2 y'(t) + y'(t). \quad (13)
\]

subject to the initial condition

\[y(0) = 1 \quad (14)\]

Taking the differential transform of (13), then

\[
\begin{align*}
(k + 1)Y(k + 1) &= -2\delta(k-1) - \delta(k-4) - Y(k) \\
&+ \sum_{k_1+l_1=0}^{k} \delta(k_1-2)Y(k_1-k_1)Y(k_1-k_1) + \sum_{k_2+l_2=0}^{k} \delta(k_2-1)Y(k_2-k_2)Y(k_2-k_2) \\
&+ \sum_{k_3+l_3=0}^{k} \delta(k_3-1)Y(k_3-k_3)Y(k_3-k_3) \quad (15)
\end{align*}
\]

From the initial condition given by Eq. (14)

\[Y(0) = 1, \quad (16)\]

is obtained. Substituting Eq. (16) into Eq. (15) and by recursive method, the results are listed follows

\[
\begin{align*}
Y(1) &= 0, \\
Y(2) &= -1, \\
Y(3) &= 0, \\
Y(4) &= 0, \\
&\vdots
\end{align*}
\]

Substituting all \(Y(k)\) into Eq. (4), we obtained the series solution as

\[y(t) = \sum_{k=0}^{\infty} Y(k)t^k = 1 - t^4. \]

Hence, the exact solution is \(1 - t^4\).

CONCLUSION

In this paper, the differential transform method (DTM) has been successfully employed to obtain analytic solution for various types of Abel equation. The method is effective, easy to use and reliable and the main benefit of the method is to offer the analytical approximation and, in many cases an exact solution, in a rapid convergent series form.

REFERENCES


