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Solution of Higher Dimensional Initial Boundary Value Problems by He's Polynomials

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Abstract: In this paper, we apply He's polynomials which are calculated from homotopy perturbation method (HPM) to solve higher dimensional initial boundary value problems with variable coefficients. The developed algorithm is quite efficient and is practically well suited for use in these problems. The proposed scheme finds the solution without any discretization, transformation or restrictive assumptions and avoids the round off errors. Several examples are given to verify the reliability and efficiency of the proposed technique.

Key words: He's polynomials · Homotopy perturbation method · Nonlinear problems · Initial value problems · Boundary value problems

INTRODUCTION

The higher dimensional initial boundary value problems [1-11] arise very frequently in mathematical physics, engineering and applied sciences. Several numerical and analytical techniques [3-11] including the spectral, characteristics and Adomian's decomposition have been developed to solve these physical problems. Most of these developed schemes have their inbuilt deficiencies. He [3, 4 12-18] developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation. In a later work Ghorbani *et al.* [19] introduced He's polynomials which are calculated from He's HPM. It has been proved [19] that He's polynomials are compatible with Adomian's polynomials but are more user friendly. The basic inspiration of this paper is the implementation of He's polynomials for higher dimensional initial boundary value problems with variable co-efficient. The fact that the proposed technique solves nonlinear problems without using the so-called Adomian's polynomials is a clear advantage of this algorithm over the decomposition method. It is worth mentioning that the homotopy perturbation method (HPM) [3,4 12-19 and 20-31] is applied without any discretization, restrictive assumption or transformation and is free from round off errors. Unlike the method of separation of variables that require initial and boundary conditions, the proposed algorithm provides an analytical solution by using the initial conditions only. The boundary conditions can be used only to justify the obtained result. The proposed method work efficiently and the results are very encouraging and reliable. Several examples are given to verify the reliability and efficiency of the suggested technique.

Homotopy Perturbation Method and He's Polynomials: To explain the homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0, \tag{1}$$

Where L is any integral or differential operator. We define a convex homotopy H (u, p) by

$$H(u, p) = (1 - p)F(u) + pL(u), \tag{2}$$

Where F (u) is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u, p) = 0, \tag{3}$$

We have

$$H(u,0) = F(u), \quad H(u,1) = L(u).$$

This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point H (v_0, 0) to a solution function H (f, 1). The embedding parameter monotonically increases from zero to unit as the trivial

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problem F (u) = 0 is continuously deforms the original problem L (u) = 0. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter [3, 4 12-19 and 20-26, 31]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [3, 4 12-19 and 20-31] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots, \tag{4}$$

if p → 1, then (4) corresponds to (2) and becomes the approximate solution of the form,

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \tag{5}$$

It is well known that series (5) is convergent for most of the cases and also the rate of convergence is dependent on L (u); see [3,4 1219 and 20-31]. We assume that (5) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to [19], He's HPM considers the solution, $u(x)$, of the homotopy equation in a series of p as follows:

$$u(x) = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + \ldots,$$

and the method considers the nonlinear term $N(u)$ as

$$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + p H_1 + p^2 H_2 + \ldots,$$

Where H_n's are the so-called He's polynomials [2, 19], which can be calculated by using the formula

$$H_n(u_0, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left(N\left(\sum_{i=0}^{n} p^i u_i\right) \right)_{p=0}, \quad n = 0, 1, 2, \ldots$$

Numerical Examples: In this section, we apply He's polynomials for solving higher dimensional initial boundary value problems with variable coefficient.

Example 3.1: Consider the two dimensional initial boundary value problem

$$u_{tt} = \frac{1}{2} y^2 u_{xx} + \frac{1}{2} x^2 u_{yy}, \qquad 0 < x, y < 1, t > 0,$$

With boundary conditions

$$u(0, y, t) = y^2 e^{-t} \qquad u(1, y, t) = (1 + y^2) e^{-t},$$

$$u(x, 0, t) = y^2 e^{-t} \qquad u(x, 1, t) = (1 + x^2) e^{-t},$$

and the initial conditions

$$u(x, y, 0) = x^2 + y^2, \qquad u_t(x, y, 0) = -(x^2 + y^2).$$

Applying the convex homotopy method, we have

$$u_0 + p u_1 + p^2 u_2 + \cdots = (x^2 + y^2) + (x^2 + y^2) t + \frac{1}{2} p \int_0^t \int_0^t y^2 \left(\frac{\partial^2 u_0}{\partial x^2} + p \frac{\partial^2 u_1}{\partial x^2} + p^2 \frac{\partial^2 u_1}{\partial x^2} + \cdots \right)^2$$

$$dt\,dt + \frac{1}{2} p \int_0^t \int_0^t x^2 \left(\frac{\partial^2 u_0}{\partial y^2} + p \frac{\partial^2 u_1}{\partial y^2} + p^2 \frac{\partial^2 u_1}{\partial y^2} + \cdots \right)^2 dt\,dt.$$

Comparing the co-efficient of like powers of p, we have

$$p^{(0)} : u_0(x, y, t) = (x^2 + y^2) - \left(x^2 + y^2\right) t,$$

$$p^{(1)} : u_1(x, y, t) = (x^2 + y^2) \frac{t^2}{2!} - (x^2 + y^2) \frac{t^3}{3!},$$

$$p^{(2)} : u_2(x, y, t) = (x^2 + y^2) \frac{t^4}{4!} - (x^2 + y^2) \frac{t^5}{5!},$$

$$p^{(3)} : u_3(x, y, t) = (x^2 + y^2) \frac{t^5}{5!} - (x^2 + y^2) \frac{t^7}{7!},$$

$$p^{(4)} : u_4(x, y, t) = (x^2 + y^2) \frac{t^8}{8!} - (x^2 + y^2) \frac{t^9}{9!},$$

$$p^{(5)} : u_5(x, y, t) = (x^2 + y^2) \frac{t^{10}}{10!} - (x^2 + y^2) \frac{t^{11}}{11!},$$

$$\vdots$$

Where p^is are the He's polynomials. The series solution is given by

$$u(x, y, t) = (x^2 + y^2) \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \frac{t^8}{8!} - \cdots \right),$$

and in a closed form by: $u(x, y, t) = (x^2 + y^2) e^{-t}$.

Example 3.2: Consider the three dimensional initial boundary value problem

$$u_{tt} = \frac{1}{45} x^2 u_{xx} + \frac{1}{45} y^2 u_{yy} + \frac{1}{45} z^2 u_{zz} - u, \quad 0 < x, y < 1, t < 0$$

Subject to the Neumann Boundary Conditions

$$u_x(0,y,z,t)=0, \qquad u_x(1,y,z,t)=6y^6z^6\sinh t, \qquad u_y(x,0,z,t)=0,$$

$$u_y(x,1,z,t)=6x^6z^6\sinh t,\ \ u_z(x,y,0,t)=0,\ \ u_z(x,y,1,t)=6x^6y^6\sinh t,$$

and the initial conditions

$$u(x,y,z,0)=0, \qquad\qquad u_t(x,y,z,0)=x^6y^6z^6.$$

Applying the convex homotopy method

$$u_0+pu_1+p^2u_2+\cdots=\left(x^6y^6z^6\right)t+\frac{1}{45}p\int_0^t\!\!\int_0^t x^2\left(\frac{\partial^2u_0}{\partial x^2}+p\frac{\partial^2u_1}{\partial x^2}+p\frac{\partial^2u_2}{\partial x^2}+\cdots\right)^2 dt\,dt$$

$$+\frac{1}{45}p\int_0^t\!\!\int_0^t\left(y^2\left(\frac{\partial^2u_0}{\partial y^2}+p\frac{\partial^2u_1}{\partial y^2}+p\frac{\partial^2u_2}{\partial y^2}+\cdots\right)^2+z^2\left(\frac{\partial^2u_0}{\partial z^2}+p\frac{\partial^2u_1}{\partial z^2}+p\frac{\partial^2u_2}{\partial z^2}+\cdots\right)^2\right)dt\,dt$$

$$-p\int_0^t\!\!\int_0^t\left(u_0+pu_1+p^2u_2+\cdots\right)dt\,dt.$$

Comparing the co-efficient of like powers of p

$$p^{(0)}:u_0(x,y,z,t)=x^6y^6z^6\,t,$$

$$p^{(1)}:u_1(x,y,z,t)=x^6y^6z^6\,\frac{t^3}{3!},$$

$$p^{(2)}:u_2(x,y,z,t)=x^6y^6z^6\,\frac{t^5}{5!},$$

$$p^{(3)}:u_3(x,y,z,t)=x^6y^6z^6\,\frac{t^7}{7!},$$

$$p^{(4)}:u_4(x,y,z,t)=x^6y^6z^6\,\frac{t^9}{9!},$$

$$\vdots$$

Where p^is are the He's polynomials. The series solution is given by

$$u(x,y,z,t)=x^6y^6z^6\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+\frac{t^9}{9!}+\cdots\right)$$

$$=x^6y^6z^6\sinh t.$$

Example 5.3: Consider the two dimensional nonlinear inhomogeneous initial boundary value problem.

$$u_{tt}=2x^2+2y^2+\frac{15}{2}(xu_{xx}^2+yu_{yy}^2), \qquad\qquad 0<x,y<1,t>0$$

With boundary conditions

$$u(0,y,t)=y^2t^2+yt^6, \qquad\qquad u(1,y,t)=(1+y^2)t^2+(1+y)t^6,$$
$$u(x,0,t)=x^2t^2+xt^6, \qquad\qquad u(x,1,t)=(1+x^2)t^2+(1+x)t^6,$$

and the initial conditions

$$u(x,y,0)=0, \quad u_t(x,y,0)=0.$$

Applying the convex homotopy method.

$$u_0+pu_1+p^2u_2+\cdots=p\int_0^t\!\!\int_0^t\left(x\left(\frac{\partial^2\tilde u_0}{\partial x^2}+p\frac{\partial^2\tilde u_1}{\partial x^2}+p^2\frac{\partial^2\tilde u_2}{\partial x^2}+\cdots\right)^2+y\left(\frac{\partial^2\tilde u_0}{\partial y^2}+p\frac{\partial^2\tilde u_1}{\partial y^2}+p^2\frac{\partial^2\tilde u_2}{\partial y^2}+\cdots\right)^2+(2x^2+2y^2)\right)dt\,dt.$$

Comparing the co-efficient of like powers of p, we have

$$p^{(0)}:u_0(x,y,t)=0,$$

$$p^{(1)}:u_1(x,y,t)=(x^2+y^2)t^2,$$

$$p^{(2)}:u_2(x,y,t)=(x+y)t^6,$$

$$p^{(3)}:u_3(x,y,t)=0,$$

$$\vdots$$

Where p^is are the He's polynomials. The solution is obtained as $u(x,y,t)=(x^2+y^2)t^2+(x+y)t^6$,

Example 5.4: Consider the three dimensional nonlinear initial boundary value problem

$$u_{tt}=(2-t^2)+u-(e^{-x}u_{xx}^2+e^{-y}u_{yy}^2+e^{-z}u_{zz}^2), \qquad 0<x,y<1,t<0$$

Subject to the Neumann Boundary Conditions:

$$u_x(0,y,z,t)=1, \qquad u_x(1,y,z,t)=e,$$

$$u_y(x,0,z,t)=0, \qquad u_y(x,1,z,t)=6x^6z^6\sinh t,$$

$$u_z(x,y,0,t)=1, \qquad u_z(x,y,1,t)=e,$$

and the initial conditions

$$u(x,y,z,0)=e^x+e^y+e^z, \qquad\qquad u_t(x,y,z,0)=0.$$

Applying the convex homotopy method, we have

$$u_0+pu_1+p^2u_2+\cdots=\left(e^x+e^y+e^z\right)-p\int_0^t\int_0^t\left(e^{-x}\left(\frac{\partial^2 u_0}{\partial x^2}+p\frac{\partial^2 u_1}{\partial x^2}+p^2\frac{\partial^2 u_2}{\partial x^2}+\cdots\right)^2-\left(2-t^2\right)\right)dtdt$$

$$-p\int_0^t\int_0^t\left(e^{-y}\left(\frac{\partial^2 u_0}{\partial y^2}+p\frac{\partial^2 u_1}{\partial y^2}+p^2\frac{\partial^2 u_2}{\partial y^2}+\cdots\right)^2+e^{-z}\left(\frac{\partial^2 u_0}{\partial z^2}+p\frac{\partial^2 u_1}{\partial z^2}+p\frac{\partial u_2}{\partial z^2}+\cdots\right)^2\right)dtdt$$

$$+p\int_0^t\int_0^t\left(u_0+pu_1+p^2u_2+\cdots\right)dtdt.$$

Comparing the co-efficient of like powers of p, we have

$$p^{(0)}: u_0(x,y,z,t) = \left(e^x+e^y+e^z\right)+t^2-\frac{t^4}{12},$$

$$p^{(1)}: u_1(x,y,z,t) = \frac{t^4}{12}-\frac{t^6}{360},$$

$$p^{(2)}: u_2(x,y,z,t) = \frac{t^6}{360}-\frac{t^8}{20160},$$

$$\vdots$$

Where p^is are the He's polynomials. The solution is obtained as $u(x,y,z,t)=\left(e^x+e^y+e^z\right)+t^2$.

CONCLUSIONS

In this paper, we use He's polynomials for solving higher dimensional initial boundary value problems with variable co-efficient. The proposed method is successfully implemented by using the initial conditions only. It is observed that the suggested algorithm is more user friendly than decomposition method.

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