On Characterization Spacelike Biharmonic Slant Helix with a Spacelike Binormal in the Lorentzian Heisenberg Group Heis³

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Abstract: In this paper, we study spacelike biharmonic slant helices in the Lorentzian Heisenberg group Heis³. We characterize position vector of spacelike biharmonic slant helices in terms of their curvature and torsion.

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INTRODUCTION

Natural scientists have long held a fascination, sometimes bordering on mystical obsession for helical structures in Nature. Helices arise in nanosprings, carbon nanotubes, α-helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA. Each strand forms a helix and two helices are held together through hydrogen bonds, ionic forces, hydrophobic interactions and van der Waals forces forming a double helix, lipid bilayers, bacterial flagella in Salmonella and E. coli, aerial hyphae in actinomyces, bacterial shape in sprochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells (helicospiral structures), [1].

On the other hand, let \(\mathcal{N}(h)\) and \(\mathcal{M}(g)\) be Riemannian manifolds. For a smooth map \(\phi: \mathcal{N} \rightarrow \mathcal{M}\), the Levi-Civita connection \(\nabla\) of \(\mathcal{N}(h)\) induces a connection \(\nabla^g\) on the pull-back bundle

\[\phi^*TM = \bigoplus_{\mathcal{N}} T\phi(p)^*M.\]

The section \(T(\phi) = \nabla^g d\phi\) is called the tension field of \(\phi\). A map \(\phi\) is said to be harmonic if its tension field vanishes identically.

A smooth map \(\phi: \mathcal{N} \rightarrow \mathcal{M}\) is said to be biharmonic if it is a critical point of the bienergy functional:

\[E_2(\phi) = \int_{\mathcal{N}} \frac{1}{2} ||T(\phi)||^2 dv_h.\]

The Euler–Lagrange equation of the bienergy is given by \(T_\phi(\phi) = 0\). Here the section \(T_\phi(\phi)\) is defined by

\[T_\phi(\phi) = -\Delta_{\phi} T(\phi) + tr R(T(\phi), d\phi)d\phi, \quad (1.1)\]

and called the bitension field of \(\phi\). The operator \(\Delta_{\phi}\) is the rough Laplacian acting on \(\Gamma(TM)\) defined by

\[\Delta_{\phi} := -\sum_{i=1}^n \left(\nabla_{\epsilon_i}^g \nabla_{\epsilon_i}^g - \nabla_{N(\phi)}^g \nabla_{\epsilon_i}^g\right).\]

Where \(\{\epsilon_i\}_{i=1}^n\) is a local orthonormal frame field of \(N\). Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study spacelike biharmonic slant helices in the Lorentzian Heisenberg group Heis³. We characterize position vector of spacelike biharmonic slant helices in terms of their curvature and torsion.

Preliminaries: The Heisenberg group Heis³ is a Lie group which is diffeomorphic to \(\mathbb{R}^3\) and the group operation is defined as:

\[(x, y, z) \ast (x', y', z') = (x + x', y + y', z + z - xy + x'y).\]

The identity of the group is \((0,0,0)\) and the inverse of \((x, y, z)\) is given by \((-x, -y, -z)\).

The left-invariant Lorentz metric on Heis³ is

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\[ g = -dx^2 + dy^2 + (xdy + dx^2). \]

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

\[
\begin{align*}
\mathbf{e}_1 &= \frac{\partial}{\partial x}, & \mathbf{e}_2 &= \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, & \mathbf{e}_3 &= \frac{\partial}{\partial z}.
\end{align*}
\]

The characterising properties of this algebra are the following commutation relations:

\[ [\mathbf{e}_2, \mathbf{e}_3] = 2\mathbf{e}_1, \quad [\mathbf{e}_3, \mathbf{e}_1] = 0, \quad [\mathbf{e}_2, \mathbf{e}_1] = 0, \]

With

\[ g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1. \]

**Proposition 2.1:** For the covariant derivatives of the Levi-Civita connection of the left-invariant metric \(g\), defined above the following is true:

\[
\nabla = \begin{pmatrix} \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix},
\]

Where the \((i,j)\)-element in the table above equals \(\nabla_{\mathbf{e}_i}\mathbf{e}_j\) for our basis

\[ \{\mathbf{e}_k, k = 1, 2, 3 \} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}. \]

We adopt the following notation and sign convention for Riemannian curvature operator:

\[ R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \]

The Riemannian curvature tensor is given by

\[ R(X, Y; Z, W) = g(R(X, Y)Z, W). \]

Moreover, we put

\[ R_{abcd} = R(\mathbf{e}_a, \mathbf{e}_b)\mathbf{e}_c R_{abcd} = R(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c, \mathbf{e}_d). \]

Where the indices \(a, b, c\) and \(d\) take the values 1, 2 and 3.

Then the non-zero components of the Riemannian curvature tensor field and of the Riemannian curvature tensor are, respectively,

\[ R_{121} = -\mathbf{e}_2, R_{131} = -\mathbf{e}_3, \quad R_{232} = 3\mathbf{e}_3, \]

and

\[ R_{2312} = -1, R_{3131} = 1, R_{2323} = -3. \]

**Spacelike Biharmonic Curves in Lorentzian Heisenberg Group Heis:** Let \(\gamma : I \rightarrow \text{Heis}^3\) be a non geodesic spacelike curve on the Lorentzian Heisenberg group \(\text{Heis}^3\) parametrized by arc length. Let \(\{T, N, B\}\) be the Frenet frame fields tangent to the Lorentzian Heisenberg group \(\text{Heis}^3\) along \(\gamma\) defined as follows:

\[ T \text{ is the unit vector field } \gamma' \text{ tangent to } \gamma, \quad N \text{ is the unit vector field in the direction of } V_{\gamma'}(\text{normal to } \gamma) \text{ and } B \text{ is chosen so that } \{T, N, B\} \text{ is a positively oriented orthonormal basis}. \]

Then, we have the following Frenet formulas:

\[ \nabla_{\gamma'} T = \kappa N, \quad \nabla_{\gamma'} N = \kappa' T + \tau B, \quad \nabla_{\gamma'} B = \tau N, \]

Where \(k\) is the curvature of \(\gamma\), \(\tau\) is its torsion and

\[ g(T, T) = 1, g(N, N) = -1, g(B, B) = 1, \]

\[ g(T, N) = g(T, B) = g(N, B) = 0. \]

If we write this curve in the another parametric representation \(\gamma = \gamma(\theta)\), where \(\theta = \int c(s) ds\). We have new Frenet equations as follows:

\[ \nabla_{T(\theta)} T(\theta) = N(\theta), \quad \nabla_{T(\theta)} N(\theta) = f(\theta) T(\theta) + B(\theta), \]

Where

\[ f(\theta) = \frac{\tau(\theta)}{\kappa(\theta)}. \]

With respect to the orthonormal basis \(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\) we can write

\[ T(\theta) = T_1(\theta)\mathbf{e}_1 + T_2(\theta)\mathbf{e}_2 + T_3(\theta)\mathbf{e}_3, \]

\[ N(\theta) = N_1(\theta)\mathbf{e}_1 + N_2(\theta)\mathbf{e}_2 + N_3(\theta)\mathbf{e}_3, \]

\[ B(\theta) = T(\theta) \times N(\theta) = B_1(\theta)\mathbf{e}_1 + B_2(\theta)\mathbf{e}_2 + B_3(\theta)\mathbf{e}_3. \]

**Theorem 3.1:** \(\gamma = \gamma(\theta)\) is a unit speed spacelike biharmonic curve in \(\text{Heis}^3\) if and only if

\[ f^2(\theta) = -2 + 4R^2(\theta), \quad f^1(\theta) = -2N_1(\theta)B_1(\theta). \]
Proof: Using Eq. (1.1) and Eq. (3.1), we have

\[ T_2(\gamma) = \nabla \mathbf{T}(\theta) + R(\mathbf{T}(\theta), \nabla \mathbf{T}(\theta)) = (1 + f^{2}(\theta)) N(\theta) + f^{\prime}(\theta) \mathbf{B}(\theta) - R(\mathbf{T}(\theta), N(\theta)) = 1 + f^{2}(\theta) N(\theta), \quad (3.4) \]

\[ f^{\prime}(\theta) = R(\mathbf{T}(\theta), N(\theta), \mathbf{B}(\theta)). \]

A direct computation using Eq. (2.4), yields

\[ R(\mathbf{T}(\theta), N(\theta), \mathbf{B}(\theta)) = 2N_{1}(\theta) B_{1}(\theta), \quad (3.5) \]

These, together with Eq. (3.4), complete the proof of the theorem.

**Theorem 3.2:** Let \( \gamma = \gamma(\theta) \) is a unit speed spacelike curve in \( \mathbb{R}^{3} \). If \( N_{1} \) constant, then \( \gamma \) is not a slant helix.

**Proof:** Assume that \( \gamma \) is a spacelike slant helix, i.e. the principal normal vector \( N \) makes a constant space-like angle, \( \Psi = \arccos \nu \), with the constant space-like vector called the axis of the slant helix. So, without loss of generality, we take the axis of a slant helix is parallel to the space-like vector \( e_{1} \). Then

\[ g(N(\theta), e_{1}) = n. \quad (3.6) \]

We use the first equation of Eq. (3.2) to get

\[ g(N(\theta), e_{1}) = N_{1}(\theta) = N_{1} = \text{constant.} \quad (3.7) \]

We have a contradiction that is \( N_{1} \) constant. Therefore \( \gamma \) is not a slant helix.

**Corollary 3.3:** Let \( \gamma = \gamma(\theta) \) is a unit speed spacelike biharmonic slant helix. Then,

\[ N_{1}(\theta) = \text{constant.} \quad (3.8) \]

**Corollary 3.4:** Let \( \gamma = \gamma(\theta) \) is a unit speed spacelike biharmonic slant helix. Then,

\[ B_{1}(\theta) = -\frac{1}{f(\theta)}[N_{1}(\theta) + \xi_{0}]. \quad (3.9) \]

Where \( \xi_{0} \) is constants of integration.

**Proof:** Differentiating Eq. (3.6) with respect to the variable \( \theta = \int \sigma(s)ds \) and using the new Frenet equations (3.1), we get

\[ g(T(\theta) + f(\theta) B(\theta), e_{1}) = 0. \quad (3.10) \]

Using Eq. (3.6), we have

\[ T_{1}(\theta) + f(\theta) B_{1}(\theta), e_{1} = 0. \quad (3.11) \]

We can use Eq. (2.3) to compute the covariant derivative of the vector fields \( T \) as

\[ \nabla \mathbf{T}(\theta) = T_{1}(\theta) e_{1} + (T_{2}(\theta) + 2T_{1}(\theta) T_{2}(\theta)) e_{2} + 2T_{1}(\theta) T_{2}(\theta) e_{3}. \]

It follows that the first components of these vectors are given by

\[ g(\nabla \mathbf{T}(\theta), e_{1}) = T_{1}(\theta). \quad (3.12) \]

On the other hand, using Frenet formulas (3.1), we have

\[ g(\nabla \mathbf{T}(\theta), e_{1}) = N_{1}. \quad (3.13) \]

These, together with Eq. (3.12) and Eq. (3.13), give

\[ T_{1}(\theta) = N_{1} = \text{constant.} \quad (3.14) \]

Integrating the above equation, we get

\[ T_{1}(\theta) = N_{1}(\theta) + \xi_{0}. \quad (3.15) \]

Where \( \gamma = \gamma(\theta) \) is constant of integration.

Substituting Eq. (3.15) in Eq. (3.11), we have Eq. (3.9).

**Lemma 3.5:** Let \( \gamma = \gamma(\theta) \) is a unit speed spacelike biharmonic slant helix. Then,

\[ N(\theta) = N_{1} e_{1} + \sqrt{1 + N_{1}^{2} \sinh \mu(\theta) e_{2} + \sqrt{1 + N_{1}^{2} \cosh \mu(\theta) e_{3}}. \quad (3.16) \]

**Proof:** Using Eq. (3.2) and Eq. (3.8), the principal normal vector can be written in the following form
\[ N(\theta) = N_1 e_1 + N_2(\theta) e_2 + N_3(\theta) e_3 \]

On the other hand the principal normal vector \( N \) is a unit timelike vector, so the following condition is satisfied
\[ N_2^2(\theta) - N_3^2(\theta) - 1 + N_1^2. \]  
(3.17)

The general solution of Eq. (3.17) can be written in the following form
\[ N_3(\theta) = \sqrt{1 + N_1^2} \cosh \mu(\theta), \]
\[ N_2(\theta) = \sqrt{1 + N_1^2} \sinh \mu(\theta), \] 
(3.18)

Where \( \mu \) is an arbitrary function of \( \theta \).
Therefore, the principal normal vector takes form Eq. (3.16).

**Theorem 3.6:** The position vector \( \gamma(\theta) \) of a spacelike biharmonic slant helix is
\[ \gamma(\theta) = \frac{N_1^2}{2} \theta^2 + c_1 \theta + c_2 \] 
\[ + \sqrt{1 + N_1^2} \left[ \int \cosh \mu(\theta) d\theta \right] d\theta e_2 \]
\[ + \sqrt{1 + N_1^2} \left[ \int \sinh \mu(\theta) d\theta \right] d\theta e_3, \]  
(3.19)

Where \( c_1, c_2 \) are constants of integration.

**Proof:** Using Lemma 3.5, we have Eq. (3.19).

**REFERENCES**

5. Happel, J. and H. Brenner, Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media, Prentice-Hall, New Jersey,