On the Behavior of Solutions of the System of Rational Difference Equations

\[ x_{n+1} = \frac{x_{n-1}}{y_n x_n - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_n - 1} \]

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Abstract: In this paper, we investigate the solutions of the system of difference equations

\[ x_{n+1} = \frac{x_{n-1}}{y_n x_n - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_n - 1} \]

Where \( y_0, y_1, x_0, x_1 \in \mathbb{R} \)

Key words: Difference equation · Difference equation systems · Solutions

INTRODUCTION

Recently, there has been great interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real-life situations in population biology, economic, probability theory, genetics, psychology etc. There are many papers with related to the difference equations system for example,

In [1] Cinar studied the solutions of the systems of the difference equations.

\[ x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}} \]

In [2] Papaschinopoulos and Schinas studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations

\[ x_{n+1} = A + \frac{y_n}{x_{n-p}}, \quad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n = 0, 1, 2, \ldots, p, q. \]

In [3] Papaschinopoulos and Schinas proved the boundedness, persistence, the oscillatory behavior and the asymptotic behavior of the positive solutions of the system of difference equations.

\[ x_{n+1} = \sum_{i=0}^{k} \frac{A_i}{x_{n-i}}, \quad y_{n+1} = \sum_{i=0}^{k} \frac{B_i}{y_{n-i}} \]

In [4, 5] Özban studied the positive solutions of the system of rational difference equations

\[ x_n = \frac{a}{y_{n-3}}, \quad y_{n+1} = \frac{b y_{n-3}}{x_{n-3} y_{n-q}} \]

\[ x_{n+1} = \frac{a}{y_{n-k}}, \quad y_{n+1} = \frac{y_n}{x_{n-m} y_{n-m-k}} \]

In [6, 7] Clark and Kulenović investigate the global asymptotic stability

\[ x_{n+1} = \frac{x_n}{a + cy_n}, \quad y_{n+1} = \frac{y_n}{b + dx_n} \]

In [8] Camouzis and Papaschinopoulos studied the global asymptotic behavior of positive solutions of the system of rational difference equations.

\[ x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}} \]

In [9] Yang, Liu and Bai considered the behavior of the positive solutions of the system of the difference equations

\[ x_n = \frac{a}{y_{n-p}}, \quad y_n = \frac{b y_{n-p}}{x_{n-q} y_{n-q}} \]

In [10] Kulenović, Nurkanović studied the global asymptotic behavior of solutions of the system of difference equations.

\[ x_{n+1} = \frac{a + x_n}{b + y_n}, \quad y_{n+1} = \frac{c + y_n}{d + z_n}, \quad z_{n+1} = \frac{e + z_n}{f + x_n} \]

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In [11] Zhang, Yang, Megson and Evans investigated the behavior of the positive solutions of the system of difference equations.

\[ x_{n+1} = A + \frac{1}{y_{n-p}}, \quad y_{n+1} = A + \frac{y_{n-1}}{x_{n-1} y_{n-2}} \]

In [12] Zhang, Yang, Evans and Zhu studied the boundedness, the persistence and global asymptotic stability of the positive solutions of the system of difference equations.

\[ x_{n+1} = A + \frac{y_{n-m}}{x_n}, \quad y_{n+1} = A + \frac{x_{n-m}}{y_n} \]

In [13] Yalcinkaya and Cinar studied the global asymptotic stability of the system of difference equations.

\[ x_{n+1}^{(1)} = \frac{x_n^{(1)} + x_n^{(2)}}{s_n^{(1)}}, \quad x_{n+1}^{(2)} = \frac{x_n^{(2)} + x_n^{(3)}}{s_n^{(2)}}, \quad x_{n+1}^{(3)} = \frac{x_n^{(3)} + x_n^{(4)}}{s_n^{(3)}}, \quad \ldots \]

\[ x_{n+1}^{(p)} = \frac{x_n^{(p)} + x_n^{(p+1)}}{s_n^{(p)}}, \quad x_{n+1}^{(p+1)} = \frac{x_n^{(p+1)} + x_n^{(p+2)}}{s_n^{(p+1)}}, \quad \ldots \]

In [14] Yalcinkaya, Cinar and Atalay investigated the solutions of the system of difference equations.

\[ x_{n+1} = \frac{x_n^{(1)}}{x_n^{(2)} - 1}, \quad x_{n+1}^{(2)} = \frac{x_n^{(3)}}{x_n^{(4)} - 1}, \quad \ldots \]

In [15] Yalcinkaya studied the global asymptotic stability of the system of difference equations.

\[ x_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \quad t_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}} \]

In [16] Iričanin and Stević studied the positive solutions of the system of difference equations.

\[ x_{n+1}^{(1)} = \frac{x_n^{(1)} + x_n^{(2)}}{s_n^{(1)}}, \quad x_{n+1}^{(2)} = \frac{x_n^{(2)} + x_n^{(3)}}{s_n^{(2)}}, \quad x_{n+1}^{(3)} = \frac{x_n^{(3)} + x_n^{(4)}}{s_n^{(3)}}, \quad \ldots \]

Also see [15, 17-20].

In this paper, we investigate the behavior of the solutions of the difference equation system.

\[ x_{n+1} = \frac{x_n - 1}{y_n x_n - 1}, \quad y_{n+1} = \frac{y_n - 1}{x_n y_n - 1} \quad (1.1) \]

Where the initial conditions are arbitrary real numbers. This paper was motivated by [21] and [22].

**Theorem 1:** Let \( y_0 = a, \ y_1 = b, \ x_0 = c, \ x_1 = d \) be arbitrary real numbers and let \( x_n, y_n \) be a solutions of the system (1.1). Also, assume that \( ad + 1 \) and \( cd + 1 \) than all solutions of (1.1) are

\[ x_n = \begin{cases} \frac{d}{(ad - 1)^{\frac{n-1}{2}}}, & n \text{ odd} \\ \frac{c}{(cb - 1)^{\frac{n}{2}}}, & n \text{ even} \end{cases} \quad (1.2) \]

\[ y_n = \begin{cases} \frac{b}{(cb - 1)^{\frac{n-1}{2}}}, & n \text{ odd} \\ \frac{a}{(ad - 1)^{\frac{n}{2}}}, & n \text{ even} \end{cases} \quad (1.3) \]

**Proof:** For \( n = 0, 1, 2, 3 \) we have

\[ x_1 = \frac{x_1}{y_0 x_1 - 1} = \frac{d}{ad - 1}, \quad y_1 = \frac{y_1}{x_0 y_1 - 1} = \frac{b}{cb - 1} \]
\[ x_2 = \frac{x_0}{y_1 y_0 - 1} = \frac{c}{b^{cb-1}} = c(b - 1) \]

\[ y_2 = \frac{y_0}{x_2 y_0 - 1} = \frac{a}{d^{ad-1}} = a(ad - 1) \]

\[ x_3 = \frac{x_1}{y_2 x_1 - 1} = \frac{d}{a^{(ad-1)^2}} = d \]

\[ y_3 = \frac{y_1}{x_3 y_1 - 1} = \frac{b}{c^{(cb-1)^2}} = b \]

for \( n = k \) assume that

\[ x_{2k-1} = \frac{d}{(ad-1)^k} \]

\[ x_{2k} = c(cb-1)^k \]

and

\[ y_{2k} = a(ad-1)^k \]

are true. Then for \( n = k+1 \) we will show that (1.2) and (1.3) are true. From (1.1), we have

\[ x_{2k+1} = \frac{x_{2k}}{y_{2k} y_{2k-1} - 1} = \frac{d}{(ad-1)^k} \frac{d}{a^{(ad-1)^2}} = \frac{d}{(ad-1)^k + 1} = \frac{d}{(ad-1)^{k+1}} \]

Also, similarly from (1.1), we have

\[ y_{2k+1} = \frac{y_{2k-1}}{x_{2k} y_{2k-1} - 1} = \frac{b}{c^{(cb-1)^k}} = \frac{b}{(cb-1)^k + 1} \]

Also, we have

\[ x_{2k+2} = \frac{x_{2k}}{y_{2k+1} y_{2k} - 1} = \frac{c(cb-1)^k}{b^{cb-1} c^{(cb-1)^k}} = \frac{c(cb-1)^k}{cb - cb + 1} = c(cb-1)^{k+1} \]

\[ y_{2k+2} = \frac{y_{2k}}{x_{2k+1} y_{2k+1} - 1} = \frac{a(ad-1)^k}{(ad-1)^{k+1} a^{(ad-1)^k}} = \frac{a(ad-1)^k}{ad - ad + 1} = a(ad-1)^{k+1} \]

\[ \square \]
Theorem 2. Let \(y_0 = a, x_0 = b, x_1 = c, x_2 = d\) be arbitrary real numbers and let \(\{x_n, y_n\}\) be a solutions of the system (1.1). If \(0 < a, b, c, d < 1\) then we have
\[
\lim_{n \to \infty} x_{2n-1} = \lim_{n \to \infty} y_{2n-1} = \pm \infty
\]
and
\[
\lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} y_{2n} = 0
\]

Proof: From \(0 < a, b, c, d < 1\) we have \(0 < ad < 1 \to 10\) and \(0 < cb < 1 \to 1 < cb - 1 < 0\). Hence, we obtain
\[
\lim_{n \to \infty} x_{2n-1} = \lim_{n \to \infty} \frac{d}{(ad - 1)^n} = d \lim_{n \to \infty} \frac{1}{(ad - 1)^n} = d, \infty = \begin{cases} -\infty, & n \text{ odd} \\ +\infty, & n \text{ even} \end{cases}
\]
and
\[
\lim_{n \to \infty} y_{2n-1} = \lim_{n \to \infty} \frac{b}{(cb - 1)^n} = b \lim_{n \to \infty} \frac{1}{(cb - 1)^n} = b, \infty = \begin{cases} -\infty, & n \text{ odd} \\ +\infty, & n \text{ even} \end{cases}
\]

Similarly, we have
\[
\lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} c(cb - 1)^n = c \lim_{n \to \infty} (cb - 1)^n = c, 0 = 0
\]
and
\[
\lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} a(ad - 1)^n = a \lim_{n \to \infty} (ad - 1)^n = a, 0 = 0
\]

Theorem 3. Let \(y_0 = a, x_0 = b, x_1 = c, x_2 = d\) be arbitrary real numbers and let \(\{x_n, y_n\}\) be a solutions of the system (1.1). If \(1 < ad, cb < 2\) then we have
\[
\lim_{n \to \infty} x_{2n-1} = \lim_{n \to \infty} y_{2n-1} = \pm \infty
\]
and
\[
\lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} y_{2n} = 0
\]

Proof: From \(1 < ad, cb < 2\) we have \(1 < ad < 2 \to 0 < ad - 1 < 1\) and \(1 < cb < 2 \to 0 < cb - 1 < 1\). Hence, we obtain
\[
\lim_{n \to \infty} x_{2n-1} = \lim_{n \to \infty} \frac{d}{(ad - 1)^n} = d \lim_{n \to \infty} \frac{1}{(ad - 1)^n} = d, \infty = \begin{cases} -\infty, & d < 0 \\ +\infty, & d > 0 \end{cases}
\]
and
\[
\lim_{n \to \infty} y_{2n-1} = \lim_{n \to \infty} \frac{b}{(cb - 1)^n} = b \lim_{n \to \infty} \frac{1}{(cb - 1)^n} = b, \infty = \begin{cases} -\infty, & b < 0 \\ +\infty, & b > 0 \end{cases}
\]

Similarly, we have
\[
\lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} c(cb - 1)^n = c \lim_{n \to \infty} (cb - 1)^n = c, 0 = 0
\]
and
\[
\lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} a(ad - 1)^n = a \lim_{n \to \infty} (ad - 1)^n = a, 0 = 0
\]

Theorem 4: Let \(\{x_n, y_n\}\) be a solutions of (1.1). If \(ad, cb \in (-\infty, -1)\) and \(ad, cb \in (2, +\infty)\) then we have
\[
\lim_{n \to \infty} x_{2n-1} = \lim_{n \to \infty} y_{2n-1} = 0
\]
and
\[
\lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} y_{2n} = \infty
\]

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Proof: From $-\infty < ad < 0 \Rightarrow -\infty < ad - 1 < -1$ we have

$$
\lim_{n \to \infty} (ad - 1)^n =
\begin{cases}
  -\infty, & n - \text{odd} \\
  +\infty, & n - \text{even}
\end{cases}
$$

and from $-\infty < cb < 0 \Rightarrow -\infty < cb - 1 < -1$ we have

$$
\lim_{n \to \infty} (cb - 1)^n =
\begin{cases}
  -\infty, & n - \text{odd} \\
  +\infty, & n - \text{even}
\end{cases}
$$

$$
\lim_{n \to \infty} x_{2n-1} = \lim_{n \to \infty} \frac{d}{(ad - 1)^n} = d \lim_{n \to \infty} \frac{1}{(ad - 1)^n} = d \cdot 0 = 0
$$

and

$$
\lim_{n \to \infty} y_{2n-1} = \lim_{n \to \infty} \frac{b}{(cb - 1)^n} = b \lim_{n \to \infty} \frac{1}{(cb - 1)^n} = b \cdot 0 = 0
$$

Similarly, we have

$$
\lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} c (cb - 1)^n = c \lim_{n \to \infty} (cb - 1)^n = c \cdot \infty =
\begin{cases}
  -\infty, & c > 0 \text{ and } n - \text{odd} \\
  +\infty, & c < 0 \text{ and } n - \text{odd} \\
  +\infty, & c > 0 \text{ and } n - \text{even} \\
  -\infty, & c < 0 \text{ and } n - \text{even}
\end{cases}
$$

and

$$
\lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} a (ad - 1)^n = a \lim_{n \to \infty} (ad - 1)^n = a \cdot \infty =
\begin{cases}
  -\infty, & a > 0 \text{ and } n - \text{odd} \\
  +\infty, & a < 0 \text{ and } n - \text{odd} \\
  +\infty, & a > 0 \text{ and } n - \text{even} \\
  -\infty, & a < 0 \text{ and } n - \text{even}
\end{cases}
$$

Theorem 5: Let $\{x_n,y_n\}$ be a solutions of (1.1). If $a,b,c,d \in \mathbb{R}$ and $-1 < a,b,c,d < 0$ then we have

$$
\lim_{n \to \infty} x_{2n-1} = \lim_{n \to \infty} y_{2n-1} = \infty
$$

and

$$
\lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} y_{2n} = 0
$$

Proof: From $-1 < a,b,c,d < 0$ we obtain $0 < ad < 1 \Rightarrow -1 < ad - 1 < 0$ and $0 < cb < 1 \Rightarrow -1 < cb - 1 < 0$ and we have

$$
\lim_{n \to \infty} x_{2n-1} = \lim_{n \to \infty} \frac{d}{(ad - 1)^n} = d \lim_{n \to \infty} \frac{1}{(ad - 1)^n} = d \cdot \infty =
\begin{cases}
  +\infty, & n - \text{odd} \\
  -\infty, & n - \text{even}
\end{cases}
$$

and

$$
\lim_{n \to \infty} y_{2n-1} = \lim_{n \to \infty} \frac{b}{(cb - 1)^n} = b \lim_{n \to \infty} \frac{1}{(cb - 1)^n} = b \cdot \infty =
\begin{cases}
  +\infty, & n - \text{odd} \\
  -\infty, & n - \text{even}
\end{cases}
$$

Similarly, we have

$$
\lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} c (cb - 1)^n = c \lim_{n \to \infty} (cb - 1)^n = c \cdot 0 = 0
$$

and

$$
\lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} a (ad - 1)^n = a \lim_{n \to \infty} (ad - 1)^n = a \cdot 0 = 0
$$
REFERENCE

1. Çinar, C., 2004. On the positive solutions of the difference equation system
   \[ x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}. \]


4. Özbak, A.Y., 2007. On the system of rational difference equations
   \[ x_n = \frac{a}{y_{n-3}}, \quad y_{n+1} = \frac{b_{n-3}}{x_{n-3}y_{n-3}}. \]

5. Özbak, A.Y., 2006. On the positive solutions of the system of rational difference equations
   \[ x_{n+1} = \frac{a}{y_{n-1}}, \quad y_{n+1} = \frac{y_{n}}{x_{n-1}y_{n-1}}. \]


   Global asymptotic behavior of a two-dimensional difference equation modelling competition.
   Nonlinear Analysis, 52: 1765-1776.

   System of Rational Difference Equations
   \[ x_{n+1} = 1 + \frac{x_n}{y_{n-1}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-1}}. \]

   \[ x_n = \frac{a}{y_{n-3}}, \quad y_n = \frac{b_{n-3}}{x_{n-3}y_{n-3}}. \]


    \[ x_n = A + \frac{1}{y_{n-1}}, \quad y_n = A + \frac{y_{n-1}}{x_{n-1}y_{n-1}}. \]
    Applied Mathematics and Computation, 176: 403-408.

    \[ x_{n+1} = A + \frac{y_{n-m}}{y_n}, \quad y_{n+1} = A + \frac{x_{n-m}}{x_n}. \]

    \[ x_{n+1} = \frac{a_{n-1}}{x_n + a_{n-1}}, \quad y_{n+1} = \frac{b_{n-1}}{y_n + b_{n-1}}. \]
    Fasciculi Mathematici, 43: 171-180.


17. Yang, X., 2005. On the system of rational difference equations
    \[ x_n = A + \frac{y_{n-1}}{y_n}, \quad y_n = A + \frac{x_{n-1}}{x_n}. \]
    Applied Mathematics and Computation, 176: 403-408.

    equation \[ x_{n+1} = \max \left( \frac{1}{x_{n-1}}, x_n \right) \]

    \[ x_{n+1} = \max \left( \frac{1}{x_n}, y_n \right), \quad y_{n+1} = \max \left( \frac{1}{y_n}, y_n \right) \]
    Denklem Sistemlerinin Çözümleri Üzerine. Ahmet Keleşoğlu
    Eğitim Fakültesi Dergisi, 28: 91-104.

    \[ x_{n+1} = \max \left( \frac{A}{y_n}, y_n \right), \quad y_{n+1} = \max \left( \frac{A}{x_n}, x_n \right) \]

    \[ x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}. \]
    Applied Mathematics and Computation, 158: 813-816.
22. Çinar, C., 2004. On the solutions of the difference equation \( x_{n+1} = \frac{x_n - 1}{1 + ax_n x_{n-1}} \). Applied Mathematics and Computation, 158: 793-797.
