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# Discrete Fourier Transform and Trigonometric Interpolation

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**Abstract:** For equidistant data points, we show how to construct the corresponding trigonometric interpolation via the Discrete Fourier Transform. We also indicate that the partial fraction expansion for the hyperbolic cotangent function can be obtained via interpolation of the Fourier transform.

**Key words:** DFT - Trigonometric interpolation - Uniform sampling - Riemann zeta function - Wallis product - Sondow's expression for $\pi$.

## INTRODUCTION

Here we realize a uniform sampling of $f(x)$ in the following $N = 2n$ equidistant data points:

$$x_j = -\pi + j\,\frac{2\pi}{N}\,, \qquad f(x_j) = f_j\,, \qquad j = 0, 1, \ldots, N-1, \tag{1}$$

Then in these region the corresponding function accepts the trigonometric approximation [1-4]:

$$f(x) = \frac{1}{2}\,a_0 + \sum_{k=1}^{n-1}\,[\,a_k\,Cos(kx) + b_k\,Sin(kx)] + \frac{1}{2}\,a_n\,, \tag{2}$$

Such that:

$$a_j = \frac{(-1)^j}{n}\sum_{k=0}^{N-1} f(x_k)\;Cos\left(kj\,\frac{\pi}{n}\right), \qquad 0 \le j \le n\,,$$
$$b_r = \frac{(-1)^r}{n}\sum_{k=1}^{N-1} f(x_k)\;Sin\left(kr\,\frac{\pi}{n}\right), \qquad 1 \le r \le n-1\,. \tag{3}$$

It is interesting note that this trigonometric interpolation has not the instability of Lagrangian interpolation of equidistant data [3, 4].

The coefficients in the harmonic expansion (2) can be calculated directly from the expressions (3), however, here we want to determine them via the Discrete Fourier Transform [DFT]. In fact, we shall indicate the essential procedure, first we obtain the values:

$$\hat{f}_k = [\,-\exp\left(\frac{ik\pi}{n}\right)]^{n-1}\,f_k\,, \qquad k = 0, 1, \ldots, N-1, \tag{4}$$

and we implement the DFT for the sequence $\{\hat{f}_0, \ldots, \hat{f}_{N-1}\}$, that is, we construct the quantities $\{F_0, \ldots, F_{N-1}\}$ via the relation [5, 6]:

$$F_k = \frac{1}{\sqrt{N}}\sum_{r=0}^{N-1} \hat{f}_r \exp\left(-\frac{ikr\pi}{n}\right), \qquad 0 \le k \le N-1, \tag{5}$$

Then the coefficients of the expansion (2) are given by:

$$a_0 = (-1)^{n-1}\sqrt{\frac{2}{n}}\;F_{n-1}\,, \qquad\qquad a_n = -\,\frac{1}{\sqrt{N}}\,F_{N-1}\,, \tag{6}$$

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$$a_r = \frac{(-1)^{n-1+r}}{\sqrt{N}} \left(F_{n-1-r} + F_{n-1+r}\right), \quad b_r = i\, \frac{(-1)^{n+r}}{\sqrt{N}} \left(F_{n-1-r} - F_{n-1+r}\right), \quad 1 \le r \le n-1.$$

Thus the DFT with uniform sampling gives a trigonometric interpolation for equidistant data points. The values $F_k$ can be determined with (5) or via the Fast Fourier Transform technique [7, 8], the FFT is merely an efficient means of computing the DFT, then it is convenient that $N = 2^m$.

For example, for $N = 4,\ n = 2$:

$$\hat{f}_0 = -f_0, \quad \hat{f}_1 = -i\, f_1, \quad \hat{f}_2 = f_2, \quad \hat{f}_3 = i\, f_3, \quad F_0 = \frac{1}{2}\left(\hat{f}_0 + \hat{f}_1 + \hat{f}_2 + \hat{f}_3\right),$$

$$F_1 = \frac{1}{2}\left[\hat{f}_0 - \hat{f}_2 - i\left(\hat{f}_1 - \hat{f}_3\right)\right], \quad F_2 = \frac{1}{2}\left[\hat{f}_0 + \hat{f}_2 - \hat{f}_1 - \hat{f}_3\right], \quad F_3 = \frac{1}{2}\left[\hat{f}_0 - \hat{f}_2 + i\left(\hat{f}_1 - \hat{f}_3\right)\right], \tag{7}$$

$$a_0 = -F_1 = \frac{1}{2}\left[f_0 + f_1 + f_2 + f_3\right], \quad a_1 = \frac{1}{2}\left(F_2 + F_0\right) = \frac{1}{2}\left(f_2 - f_0\right),$$

$$a_2 = -\frac{1}{2} F_3 = \frac{1}{4}\left(f_0 + f_2 - f_1 - f_3\right), \quad b_1 = \frac{i}{2}\left(F_2 - F_0\right) = \frac{1}{2}\left(f_3 - f_1\right),$$

In agreement with the expressions (3) of trigonometric interpolation.

**Remark 1:** The inversion of (6) implies the relations:

$$F_{n-1} = (-1)^{n-1} \sqrt{\frac{n}{2}}\, a_0, \qquad F_{N-1} = -\sqrt{N}\, a_n,$$

$$F_{n-1-r} = \frac{(-1)^{n-1+r}}{2} \sqrt{N}\ \left(a_r + i\, b_r\right), \qquad F_{n-1+r} = \frac{(-1)^{n-1+r}}{2} \sqrt{N}\ \left(a_r - i\, b_r\right) = F^*_{n-1-r}, \tag{8}$$

For $1 \le r \le n-1$, which give us the DFT if we know the corresponding harmonic interpolation (2). Thus we see the relationship between the trigonometric interpolation of equidistant data points and the Discrete Fourier Transform.

Mehta-Zhu [9] studied the following three expressions:

$$sin(\pi\, \tau) = \pi\, \tau\, \prod_{n=1}^{\infty} \left(1 - \frac{\tau^2}{n^2}\right), \tag{9}$$

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{2\,\pi^2} \sum_{n=1}^{\infty} \frac{1}{\left(\frac{x}{2\pi}\right)^2 + n^2}, \tag{10}$$

$$\pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}}\, \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s), \tag{11}$$

The infinite product expansion for the sine function, the partial fraction expansion for the hyperbolic cotangent function and the functional equation for the Riemann zeta function [10, 11], respectively, and they showed the interesting result:

$$(9) \quad \Leftrightarrow \quad (10) \quad \Leftrightarrow \quad (11), \tag{12}$$

That is, there is equivalence between these three relations. Here we wish indicate that (10) can be obtained via interpolation of the Fourier transform, in fact, Lanczos [4] used this technique to deduce the property:

$$cot\left(\frac{y}{2}\right) = \frac{2}{y} + \frac{y}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{\left(\frac{y}{2\pi}\right)^2 - n^2}, \tag{13}$$

Which for $y = i\, x$ implies the following expansion for the hyperbolic cotangent:

$$coth\left(\frac{x}{2}\right) = \frac{2}{x} + \frac{x}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(\frac{x}{2\pi})^2 + n^2}, \tag{14}$$

Finally, (10) is immediate because $\frac{x}{e^x - 1} = \frac{x}{2}\, coth\left(\frac{x}{2}\right) - \frac{x}{2}$.

**Remark 2:** It is possible to obtain (13) if we take the natural logarithm of (9) and after we apply $\frac{d}{d\tau}$. The inverse of (14) is given by [4]:

$$tanh\left(\frac{x}{2}\right) = \frac{4\,x}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(\frac{x}{\pi})^2 + (2n+1)^2}. \tag{15}$$

Lanczos [4] used interpolation techniques to obtain the expressions:

$$\cot(\pi x) = \frac{2}{\pi}x\left(\frac{1}{2x^2} - \frac{1}{1-x^2} - \frac{1}{4-x^2} - \frac{1}{9-x^2} - \ \dots\right), \quad \tan\left(\frac{\pi}{2}x\right) = \frac{4}{\pi}x\left(\frac{1}{1-x^2} + \frac{1}{9-x^2} + \frac{1}{25-x^2} + \ \dots\right), \tag{16}$$

Therefore:

$$\sum_{k=1}^{\infty}\frac{1}{4\,k^2-x^2} = \frac{1}{2\,x^2} - \frac{\pi}{4\,x}\left[2\cot(\pi x) + \tan\left(\frac{\pi}{2}x\right)\right] = \frac{2-\pi\, x\, cot(\frac{\pi}{2}x)}{4\,x^2}. \tag{17}$$

From (17) when $x \longrightarrow 0$ and the Bernoulli-Hôpital rule we deduce the following value of the Riemann zeta function [10-12]:

$$\zeta(2) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}; \tag{18}$$

and (17) with $x = 1$ implies the formula:

$$\sum_{k=1}^{\infty}\frac{1}{4\,k^2-1} = \frac{1}{2}, \tag{19}$$

Which is a telescoping sum [13] because:

$$\sum_{k=1}^{\infty}\frac{1}{4\,k^2-1} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \ \dots\right) = \frac{1}{2}.$$

On the other hand, we have the Wallis product [14-16]:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{4\,n^2}{4\,n^2-1} = \prod_{n=1}^{\infty}\left(1 + \frac{1}{4\,n^2-1}\right), \tag{20}$$

Hence the interesting Sondow's expression for $\pi$ [13, 17, 18]:

$$\pi\sum_{k=1}^{\infty}\frac{1}{4\,k^2-1} = \prod_{n=1}^{\infty}\left(1 + \frac{1}{4\,n^2-1}\right); \tag{21}$$

Is consequence from (19) and (20).

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