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On the Euler's Totient Function and a Binomial Inversion

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Abstract: We exhibit a simple deduction of a result of Mathews involving the Euler's totient function as the determinant of certain matrix. Besides, we show that a recent result of Fan-Guo is a immediate application of the known binomial inversion.

Key words: Euler's totient function - Dirichlet convolution - Laguerre polynomials - Binomial inversion - Möbius function

INTRODUCTION

We shall use the notation of Sivaramakrishnan [1]. We know the following relation involving the Euler's totient function [1-5]:

$$\sum_{d|n} \varphi(d) = I(n) = n, \tag{1}$$

Which invites to consider the expression:

$$\sum_{j=1}^{n} \varphi(j)\ P\left(\frac{m}{j}\right) = m, \ \ 1 \le m \le n, \qquad P\left(\frac{k}{j}\right) = \begin{cases} 1, & if\ \ j|k \\ 0, & otherwise \end{cases} . \tag{2}$$

For example, (1) for $n = 1, 2, 3$ generates the same equations than (2) with $n = 3$ and $m = 1, 2, 3$. The system (2) can be written in matrix form:

$$\tilde{B} \begin{pmatrix} \varphi(1) \\ \varphi(2) \\ \vdots \\ \varphi(n) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}, \quad \tilde{B} = \left(\tilde{B}_{ij}\right) = P\left(\frac{i}{j}\right), \quad 1 \le i, j \le n\,, \quad det\,\tilde{B} = 1, \tag{3}$$

where $\tilde{B}$ is a lower triangular matrix.

From (3) for $n = 3$ and 4:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi(1) \\ \varphi(2) \\ \varphi(3) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \Rightarrow \quad \varphi(3) = det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad \Rightarrow \quad \varphi(4) = det \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 4 \end{pmatrix}, \text{ etc.} \tag{4}$$

That is:

$$\varphi(n) = det\ \left(A_{ij}\right), \qquad A_{ij} = \begin{cases} B_{ij}\,, & 1 \le j \le n-1 \\ i, & j = n \end{cases}, \tag{5}$$

Result deduced by Mathews [1, 6].

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We have efficient methods [7] to realize the inversion of lower triangular matrices, then we multiply (4) by the corresponding $\tilde{B}^{-1}$ to obtain the expressions:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \varphi(1) \\ \varphi(2) \\ \varphi(3) \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \end{pmatrix}, \tag{6}$$

Associated with the following system involving the Möbius function [1-3, 8, 9]:

$$\sum_{j=1}^{n} j\, Q\left(\frac{m}{j}\right) = \varphi(m), \ 1 \leq m \leq n, \qquad Q\left(\frac{k}{j}\right) = \begin{cases} \mu(\frac{k}{j}), & if \ \ j|k \\ 0, & otherwise \end{cases}, \tag{7}$$

Which is equivalent to the Dirichlet inversion of (1):

$$\sum_{d|n} \mu(\tfrac{n}{d})\ d = \varphi(n). \tag{8}$$

Fan – Guo [10] obtained the following property for $m, n \geq 1$:

$$\frac{f(n)}{(n+m-1)\,!} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{g(k)}{(k+m-1)\,!} \ \Leftrightarrow\ g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(k)}{(k+m-1)\,!}. \tag{9}$$

On the hand, it is very known the binomial inversion [11, 12]:

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k \quad \Leftrightarrow \quad b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k\,, \tag{10}$$

where we can select:

$$a_n = \frac{f(n)}{(n+m-1)\,!} \qquad \text{and} \qquad b_k = \frac{g(k)}{(k+m-1)\,!}, \tag{11}$$

To obtain (9), q.e.d.

In the case $m = 1$, from (9):

$$\frac{f(n)}{n\,!} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{g(k)}{k\,!} \quad \Leftrightarrow \quad \frac{g(n)}{n\,!} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(k)}{k\,!}, \tag{12}$$

With application to Laguerre polynomials [13-15] if:

$$f(n) = n\,!\ L_n(x) \qquad \text{and} \qquad g(k) = x^k\,, \tag{13}$$

Then (12) implies the known identity [16]:

$$x^n = \sum_{k=0}^{n} (-1)^k\, n! \binom{n}{k} L_k(x). \tag{14}$$

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