Studies in Nonlinear Sciences 6 (4): 59-60, 2021
ISSN 2221-3910

DOI: 10.5829/idosi.sns.2021.59.60

Rathie-Paris Identity for the Gauss Hypergeometric Function

M. Jiménez-Hernández and J. López-Bonilla

ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: We show an alternative deduction for the Rathie-Paris formula satisfied by the Gauss hypergeometric function.

Key words: Hypergeometric function 2F1 - Incomplete Beta function

INTRODUCTION

Rathie-Paris [1] obtained the following relation for $0 < x < 1$:

$$G_n(x) \equiv x\ {}_2F_1\left(-n, 1; -2n; \frac{1}{1-x}\right) + (1-x)\ {}_2F_1\left(-n, 1; -2n; \frac{1}{x}\right) = \frac{(n!)^2}{(2n)!\,[x\,(1-x)]^n}, \qquad (1)$$

Involving the Gauss hypergeometric function. Here we show other approach towards this identity.

In [2] is the expression:

$${}_2F_1(-n, 1; c; z) = (1-c)\, z^{1-c}\, (z-1)^{c+n-1}\, B_{1-\frac{1}{z}}(1-c-n, n+1), \qquad (2)$$

With the participation of the incomplete Beta function:

$$B_w(a,b) = \int_0^w t^{a-1}\,(1-t)^{b-1}\,dt. \qquad (3)$$

Therefore:

$$G_n(x) = \frac{2n+1}{[x\,(1-x)]^n}\, H_n(x), \qquad (4)$$

where:

$$H_n(x) \equiv B_x(n+1, n+1) + B_{1-x}(n+1, n+1) = \int_0^x [t(1-t)]^n\,dt + \int_0^{1-x} [t(1-t)]^n\,dt, \qquad (5)$$

Then it is immediate that $\frac{d}{dx} H_n(x) = 0$, thus (5) is independent of x and we can calculate $H_n(x)$ with any value of x, hence we take $x = \frac{1}{2}$:

$$H_n\left(\frac{1}{2}\right) = 2\, B_{\frac{1}{2}}(n+1, n+1) = 2 \int_0^{\frac{1}{2}} [t(1-t)]^n\,dt. \qquad (6)$$

From (2) and [3]:

$$B_{\frac{1}{2}}(n+1, n+1) = \frac{1}{(2n+1)\,2^{2n+1}}\ {}_2F_1(-n, 1; -2n; 2), \qquad {}_2F_1(-n, 1; -2n; 2) = \frac{2^{2n}\,(n!)^2}{(2n)!}, \qquad (7)$$

whose application in (6) gives the expression:

$$H_n(x) = \frac{(n!)^2}{(2n+1)!}, \qquad (8)$$

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México. E-mail: jlopezb@ipn.mx

Then (4) and (8) imply (1), q.e.d.

REFERENCES

1. Rathie, A.K. and R.B. Paris, 2020. On a new result for the hypergeometric function, arXiv: 2007.09661v1 [math.CA] 19 Jul 2020.
2. https://functions.wolfram.com/HypergeometricFunctions/Hypergeometric2F1/03/06/06/
3. https://functions.wolfram.com/HypergeometricFunctions/Hypergeometric2F1/03/05/08/