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q-Hypergeometric Series

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Abstract: We study several q-hypergeometric series using the q-version of Petkovsek-Wilf-Zeilberger's method.

Key words: q-Hypergeometric series - q-Petkovsek-Wilf-Zeilberger's technique - q-analysis

INTRODUCTION

In [1] are the following q-series:

$$A \equiv \sum_{k=0}^{n} \binom{n}{k}_q (-t)^k \, q^{\binom{k}{2}} \, \frac{(b\,;q)_k}{(bt\,;q)_k} = \frac{(t\,;q)_n}{(bt\,;q)_n}, \tag{1}$$

$$B \equiv \sum_{n=0}^{\infty} \frac{(a\,;q)_n}{(e\,;q)_n\,(q\,;q)_n} \left(-\frac{e}{a}\right)^n q^{\binom{n}{2}} = \frac{(\frac{e}{a}\,;q)_\infty}{(e\,;q)_\infty}, \tag{2}$$

$$C \equiv \sum_{j=0}^{\infty} \frac{(b-q^k)\,(b-q^{k+1})\cdots(b-q^{k+j-1})}{(q\,;q)_j}\, t^j = \frac{(tq^k\,;q)_\infty}{(bt\,;q)_\infty}, \tag{3}$$

$$D \equiv \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}\, x^k}{(q\,;q)_k\,(bx\,;q)_k}\, (b-a)(b\,q-a)\cdots(b\,q^{k-1}-a) = \frac{(ax\,;q)_\infty}{(bx\,;q)_\infty}, \tag{4}$$

Here we shall show these identities using the Petkovsek-Wilf-Zeilberger's method [2-11] adapted to q-analysis [1, 12, 13].

In fact:

$$A = \sum_{k=0}^{\infty} r_k\,, \qquad \frac{r_{k+1}}{r_k} = \frac{(1-q^{-n}\,q^k)\,(1-b\,q^k)}{(1-bt\,q^k)\,(1-q^{k+1})}\, t\,q^n \qquad \therefore \qquad A = {}_2F_1(q^{-n},\ b;\ bt; q,\ tq^n), \tag{5}$$

But we have the Heine's q-Gauss summation formula [1] for this q-hypergeometric function [14, 15]:

$${}_2F_1\left(a, b; c; q, \frac{c}{ab}\right) = \frac{(\frac{c}{a}\,;q)_\infty\,(\frac{c}{b}\,;q)_\infty}{(c\,;q)_\infty\,(\frac{c}{ab}\,;q)_\infty}, \tag{6}$$

Also obtained by Jacobi and Ramanujan; then from (5) and (6):

$$A = \frac{(bt\,q^n\,;q)_\infty\,(t\,;q)_\infty}{(tq^n\,;q)_\infty\,(bt\,;q)_\infty} = \frac{(t\,;q)_n}{(bt\,;q)_n}\,, \qquad q.e.d.$$

Similarly:

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$$B = \sum_{n=0}^{\infty} s_n \ , \qquad \frac{s_{n+1}}{s_n} = \frac{(1 - a\, q^n)\,(1 - \frac{1}{Q} q^n)}{(1 - e\, q^n)\,(1 - q^{n+1})}\, \frac{Qe}{a} \qquad \therefore \qquad B = \ {}_2F_1\left(a \ , \ \frac{1}{Q} \ ; e \ ; q \ , \ \frac{Qe}{a}\right), \tag{7}$$

and in the final step we will apply $\lim_{Q \to 0}$; with (6) and (7):

$$B = \frac{(\frac{e}{a} ; q)_\infty\, (Qe \ ; q)_\infty}{(e \ ; q)_\infty\, (\frac{Qe}{a} ; q\,)_\infty} \ \underset{Q \to 0}{\longrightarrow} \ \frac{(\frac{e}{a} ; q)_\infty}{(e \ ; q)_\infty} \ , \quad q.e.d.$$

From (3):

$$C = \sum_{j=0}^{\infty} u_j \ , \qquad \frac{u_{j+1}}{u_j} = \frac{(1 - \frac{q^k}{b}\, q^j)}{(1 - q^{j+1})}\, bt \qquad \therefore \qquad C = \ {}_1F_0\left(\frac{q^k}{b} \ ; q, bt\right), \tag{8}$$

But we know the following result [1]:

$${}_1F_0(a \ ; q, z) = \frac{(az \ ; q)_\infty}{(z \ ; q)_\infty}, \tag{9}$$

Hence (8) and (9) imply (3), *q.e.d.*

From (4):

$$D = \sum_{k=0}^{\infty} v_k \ , \qquad \frac{v_{k+1}}{v_k} = \frac{\left(1 - \frac{1}{Q} q^k\right)(1 - \frac{b}{a} q^k)}{(1 - bx\, q^k)}\, Qax \qquad \therefore \qquad D = \ {}_2F_1\left(\frac{1}{Q} \ , \frac{b}{a} \ ; bx; q, Qax\right),$$

where we can apply (6) to obtain:

$$D = \frac{(Qbx \ ; q)_\infty\, (ax \ ; q)_\infty}{(Qax \ ; q)_\infty\, (bx \ ; q)_\infty} \ \underset{Q \to 0}{\longrightarrow} \ \frac{(ax \ ; q)_\infty}{(bx \ ; q)_\infty} \ , \quad q.e.d.$$

The identity (4) was deduced by Cauchy and Ramanujan. The property (2) is a particular case of the Andrews formula [1]:

$$\sum_{n=0}^{\infty} \frac{(a \ ; q)_n\, (b \ ; q)_n}{(e \ ; q)_n\, (ax \ ; q)_n\, (q \ ; q)_n} \left(-\frac{ex}{b}\right)^n q^{\binom{n}{2}} = \frac{(x \ ; q)_\infty}{(ax \ ; q)_\infty} \ {}_2F_1\left(a, \ \frac{e}{b} \ ; e; q, x\right), \tag{10}$$

For $x = \frac{b}{a}$.

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