Studies in Nonlinear Sciences 6 (4): 54-55, 2021
ISSN 2221-3910
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DOI: 10.5829/idosi.sns.2021.54.55

$(q;q)_n$ and Partial Bell Polynomials

J. López-Bonilla and M. Sedeño-Juárez

ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista 07738 CDMX, México

Abstract: We give a connection between $(q;q)_n$ and the incomplete exponential Bell polynomials.

Key words: Bell polynomials - q-analysis - Hessenberg determinant

INTRODUCTION

We know the results [1]:

$$(x;q)_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\binom{k}{2}} x^k, \qquad \frac{1}{(x;q)_{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k}_q x^k, \tag{1}$$

Then it is immediate the expression:

$$\sum_{k=0}^{\infty} \frac{1}{(q;q)_k} x^k = \frac{1}{\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} x^k}, \qquad (q;q)_0 = 1, \tag{2}$$

Therefore [2-4]:

$$\frac{1}{(q;q)_n} = \frac{1}{n!} \sum_{k=1}^{n} (-1)^k k!\ B_{n,k}\left(-\frac{1!\, q^{\binom{1}{2}}}{(q;q)_1}, \frac{2!\, q^{\binom{2}{2}}}{(q;q)_2}, -\frac{3!\, q^{\binom{3}{2}}}{(q;q)_3}, \frac{4!\, q^{\binom{4}{2}}}{(q;q)_4}, \dots, \frac{(-1)^{n-k+1} (n-k+1)!\, q^{\binom{n-k+1}{2}}}{(q;q)_{n-k+1}}\right), \tag{3}$$

Involving the partial Bell polynomials [2, 5-7], with the recurrence relation:

$$\sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k (q;q)_{n-k}} = 0, \qquad n \geq 1. \tag{4}$$

We can apply to (3) the Birmajer-Gil-Weiner's inversion process [8] to obtain:

$$\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} = \frac{1}{n!} \sum_{k=1}^{n} (-1)^k k!\ B_{n,k}\left(\frac{1!}{(q;q)_1}, \frac{2!}{(q;q)_2}, \frac{3!}{(q;q)_3}, \frac{4!}{(q;q)_4}, \dots, \frac{(n-k+1)!}{(q;q)_{n-k+1}}\right). \tag{5}$$

On the other hand, from [9, 10] we have that relations type (2) are equivalent to the following Hessenberg determinant:

$$\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} = (-1)^n \begin{vmatrix} \frac{1}{(q;q)_1} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & 0 & \cdots & 0 \\ \frac{1}{(q;q)_3} & \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 1 \\ \frac{1}{(q;q)_n} & \frac{1}{(q;q)_{n-1}} & \frac{1}{(q;q)_{n-2}} & \cdots & \cdots & \frac{1}{(q;q)_1} \end{vmatrix}. \tag{6}$$

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista 07738 CDMX, México; jlopezb@ipn.mx

The expressions (3), (5) and (6) were inspired by the formulae of Malenfant [11] and Jha [12] for the partition function [6, 7].

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