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Boost in an Arbitrary Direction

C.G. León-Vega, J. López-Bonilla and R. Meneses-González

ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 5, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: We exhibit the unimodular matrix generating, in special relativity, a boost with arbitrary direction, which is used to construct solutions of the Dirac equation for free particle.

Key words: Lorentz mapping - Unimodular matrix - Dirac 4-spinor - Boost in special relativity

INTRODUCTION

Here we consider the matrix $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\alpha\delta - \beta\gamma = 1$ for the values:

$$\alpha = \cosh\left(\frac{\phi}{2}\right) - \frac{v_z}{v}\sinh\left(\frac{\phi}{2}\right), \qquad \delta = \cosh\left(\frac{\phi}{2}\right) + \frac{v_z}{v}\sinh\left(\frac{\phi}{2}\right),$$
$$\beta = -\frac{1}{v}\left(v_x + i\,v_y\right)\sinh\left(\frac{\phi}{2}\right), \qquad \gamma = -\frac{1}{v}\left(v_x - i\,v_y\right)\sinh\left(\frac{\phi}{2}\right), \tag{1}$$

where ϕ is determined by the speed between the frames of reference:

$$\cosh\phi = \tilde{\gamma} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \sinh\phi = \frac{v}{c}\,\tilde{\gamma}, \quad v^2 = v_x^2 + v_y^2 + v_z^2, \quad \vec{p} = m\,\vec{v} = m_0\,\tilde{\gamma}\,\vec{v},$$
$$|\vec{p}| = p = m\,v, \quad E^2 = p^2c^2 + m_0^2\,c^4, \quad \cosh\left(\frac{\phi}{2}\right) = \sqrt{\frac{E + m_0\,c^2}{2\,m_0\,c^2}}, \quad \sinh\left(\frac{\phi}{2}\right) = \sqrt{\frac{E - m_0\,c^2}{2\,m_0\,c^2}}, \tag{2}$$
$$\tanh\phi = \frac{v}{c}, \quad \tanh\left(\frac{\phi}{2}\right) = \frac{\sinh\phi}{1+\cosh\phi} = \frac{p\,c}{E + m_0\,c^2}, \quad 2\sinh^2\left(\frac{\phi}{2}\right) = \cosh\phi - 1 = \tilde{\gamma} - 1,$$

Then the parameters (1) generate a proper Lorentz transformation future preserving $\tilde{x}^\mu = L^\mu{}_\lambda x^\lambda$ via the expressions [1-10]:

$$L^0{}_0 = \frac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}\right), \quad L^1{}_0 = \frac{1}{2}\left(\bar{\alpha}\,\gamma + \bar{\beta}\delta\right) + cc, \quad L^2{}_0 = -\frac{i}{2}\left(\alpha\bar{\gamma} - \bar{\beta}\delta\right) + cc,$$
$$L^0{}_1 = \frac{1}{2}\left(\bar{\alpha}\beta + \bar{\gamma}\delta\right) + cc, \quad L^1{}_1 = \frac{1}{2}\left(\bar{\alpha}\delta + \beta\bar{\gamma}\right) + cc, \quad L^2{}_1 = -\frac{i}{2}\left(\alpha\bar{\delta} + \beta\bar{\gamma}\right) + cc,$$
$$L^0{}_2 = -\frac{i}{2}\left(\bar{\alpha}\beta + \bar{\gamma}\delta\right) + cc, \quad L^1{}_2 = -\frac{i}{2}\left(\bar{\alpha}\delta + \beta\bar{\gamma}\right) + cc, \quad L^2{}_2 = \frac{1}{2}\left(\bar{\alpha}\delta - \bar{\beta}\gamma\right) + cc,$$
$$L^0{}_3 = \frac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}\right), \quad L^1{}_3 = \frac{1}{2}\left(\bar{\alpha}\gamma - \bar{\beta}\delta\right) + cc, \quad L^2{}_3 = -\frac{i}{2}\left(\alpha\bar{\gamma} + \bar{\beta}\delta\right) + cc, \tag{3}$$
$$L^3{}_0 = \frac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}\right), \quad L^3{}_1 = \frac{1}{2}\left(\bar{\alpha}\beta - \bar{\gamma}\delta\right) + cc, \quad L^3{}_2 = -\frac{i}{2}\left(\bar{\alpha}\beta - \bar{\gamma}\delta\right) + cc,$$
$$L^3{}_3 = \frac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}\right), \quad \alpha\delta - \beta\gamma = 1,$$

where *cc* means the complex conjugate of all the previous terms. Thus we obtain the following Lorentz symmetric matrix representing a boost in an arbitrary direction:

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 5, 1er. Piso, Col. Lindavista CP 07738, CDMX, México. E-mail: jlopezb@ipn.mx.

$$(L^{\mu}{}_{\lambda}) = \begin{pmatrix} \tilde{\gamma} & -\frac{v_x}{c}\tilde{\gamma} & -\frac{v_y}{c}\tilde{\gamma} & -\frac{v_z}{c}\tilde{\gamma} \\ -\frac{v_x}{c}\tilde{\gamma} & 1+\frac{v_x^2}{v^2}(\tilde{\gamma}-1) & \frac{v_x v_y}{v^2}(\tilde{\gamma}-1) & \frac{v_x v_z}{v^2}(\tilde{\gamma}-1) \\ -\frac{v_y}{c}\tilde{\gamma} & \frac{v_x v_y}{v^2}(\tilde{\gamma}-1) & 1+\frac{v_y^2}{v^2}(\tilde{\gamma}-1) & \frac{v_y v_z}{v^2}(\tilde{\gamma}-1) \\ -\frac{v_z}{c}\tilde{\gamma} & \frac{v_x v_z}{v^2}(\tilde{\gamma}-1) & \frac{v_y v_z}{v^2}(\tilde{\gamma}-1) & 1+\frac{v_z^2}{v^2}(\tilde{\gamma}-1) \end{pmatrix}, \tag{4}$$

That is [11]:

$$\tilde{t} = \tilde{\gamma}\left(t - \frac{1}{c^2}\,\vec{v}\cdot\vec{x}\right), \qquad \tilde{\vec{x}} = \vec{x} + \left[\frac{\tilde{\gamma}-1}{v^2}\,\vec{v}\cdot\vec{x} - \tilde{\gamma}\,t\right]\vec{v}\,. \tag{5}$$

On the other hand, the Dirac spinor obeys the transformation law [11, 12]:

$$\tilde{\psi}(\tilde{x}^{\mu}) = S(L)\,\psi(x^{\mu})\,. \tag{6}$$

For a non-singular matrix S such that:

$$L^{\mu}{}_{\nu}\,S\,\gamma^{\nu} = \gamma^{\mu}\,S\,, \tag{7}$$

Involving the gamma matrices. The Dirac equation for spin-1/2 particles [11-14]:

$$(i\gamma^{\mu}\partial_{\mu} - m_0)\psi = 0, \qquad i = \sqrt{-1}, \qquad \partial_{\mu} = \frac{\partial}{\partial x^{\mu}}, \tag{8}$$

where ψ is a 4-spinor with the γ^{μ} matrices verifying the anticommutator:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}\,I_{4x4}\,, \qquad (g^{\mu\nu}) = Diag(1,-1,-1,-1). \tag{9}$$

In the Dirac-Pauli (or standard) representation:

$$\gamma_D^0 = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \quad \gamma_D^j = \begin{pmatrix} O & \sigma_j \\ -\sigma_j & O \end{pmatrix}, \quad j = 1, 2, 3, \tag{10}$$

With the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{11}$$

The transformation law of ψ, under the orthochronic and proper Lorentz group is governed by the matrix [10, 15]:

$$S = \begin{pmatrix} A & C \\ C & A \end{pmatrix}, \qquad A = \frac{1}{2}\begin{pmatrix} \bar{\alpha}+\delta & \bar{\beta}-\gamma \\ \bar{\gamma}-\beta & \alpha+\bar{\delta} \end{pmatrix}, \qquad C = \frac{1}{2}\begin{pmatrix} \bar{\alpha}-\delta & \bar{\beta}+\gamma \\ \bar{\gamma}+\beta & \bar{\delta}-\alpha \end{pmatrix}, \tag{12}$$

Verifying (6); therefore, (1) and (12) imply the following matrix which allows to relate Dirac 4-spinors between frames of references connected by an arbitrary boost:

$$S = \cosh\left(\frac{\phi}{2}\right)\begin{pmatrix} I & G \\ G & I \end{pmatrix}, \quad S^{-1} = \cosh\left(\frac{\phi}{2}\right)\begin{pmatrix} I & -G \\ -G & I \end{pmatrix}, \quad G = -\frac{c}{E + m_0\,c^2}\begin{pmatrix} p_z & p_x - i\,p_y \\ p_x + i\,p_y & -p_z \end{pmatrix}, \tag{13}$$

where $p_j = mv_j,\ \ j = x, y, z.$

If a particle is at rest, then (8) admits the solutions [11]:

$$\psi_I^0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-\frac{i}{\hbar}Et}, \qquad \psi_{II}^0 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} e^{-\frac{i}{\hbar}Et}, \qquad \psi_{III}^0 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} e^{\frac{i}{\hbar}Et}, \qquad \psi_{IV}^0 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} e^{\frac{i}{\hbar}Et}, \tag{14}$$

with $E = m_0\, c^2$ and $Et = p_\mu x^\mu$. Thus, the corresponding 4-spinors for a free particle with velocity $\vec{v}$ are given by $\psi_K = S^{-1}\,\psi_K^0$, $K = I, \dots, IV,$ hence:

$$\psi_I = \cosh\left(\frac{\phi}{2}\right) \begin{pmatrix} 1 \\ 0 \\ \frac{c\, p_z}{E + m_0\, c^2} \\ \frac{c\,(p_x + i\, p_y)}{E + m_0\, c^2} \end{pmatrix} e^{-\frac{i}{\hbar}p_\mu x^\mu}, \qquad \psi_{II} = \cosh\left(\frac{\phi}{2}\right) \begin{pmatrix} 0 \\ 1 \\ \frac{c\,(p_x - i\, p_y)}{E + m_0\, c^2} \\ -\frac{c\, p_z}{E + m_0\, c^2} \end{pmatrix} e^{-\frac{i}{\hbar}p_\mu x^\mu},$$

$$\psi_{III} = \cosh\left(\frac{\phi}{2}\right) \begin{pmatrix} \frac{c\, p_z}{E + m_0\, c^2} \\ \frac{c\,(p_x + i\, p_y)}{E + m_0\, c^2} \\ 1 \\ 0 \end{pmatrix} e^{\frac{i}{\hbar}p_\mu x^\mu}, \qquad \psi_{IV} = \cosh\left(\frac{\phi}{2}\right) \begin{pmatrix} \frac{c\,(p_x - i\, p_y)}{E + m_0\, c^2} \\ -\frac{c\, p_z}{E + m_0\, c^2} \\ 0 \\ 1 \end{pmatrix} e^{\frac{i}{\hbar}p_\mu x^\mu}, \tag{15}$$

with the quantities (2), in agreement with the literature [11, 12].

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