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Sum of Divisors and Partition Functions

[1]*J. Morales,* [1]*G. Ovando and* [2]*J. López-Bonilla*

[1] CBI-Área de Física Atómica Molecular Aplicada,
Universidad Autónoma Metropolitana-Azcapotzalco, Av. San Pablo 180,
Col. Reynosa-Tamaulipas CP 02200, CDMX, México,
[2] ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: We exhibit several connections between the Euler, divisors and partition functions via the partial Bell polynomials and Hessenberg determinants.

Key words: Partitions, Bell polynomials, q-analysis, Sum of divisors, Hessenberg determinant.

INTRODUCTION

We know the following result [1-6] for the partition function $p(n)$ in terms of the Euler function:

$$\sum_{n=0}^{\infty} p(n)\, t^n = \frac{1}{\sum_{r=0}^{\infty} a_r\, t^r} = \frac{1}{E(q)} \equiv \frac{1}{(q;q)_\infty}\,, \qquad p(0) = a_0 = 1, \tag{1}$$

where:

$$a_j = \begin{cases} 0 & if \quad j \neq \frac{N}{2}\,(3N+1), \\ (-1)^N & if \quad j = \frac{N}{2}\,(3N+1), \end{cases} \qquad N = 0, \pm 1, \pm 2, \ldots\,, \tag{2}$$

Therefore [6-10]:

$$E(q) = \sum_{n=0}^{\infty} \frac{E^{(n)}(0)}{n!}\, q^n\,, \qquad E^{(n)}(0) = n!\; a_n\,, \tag{3}$$

$$p(n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k\, k!\; B_{n,k}(1!\; a_1, 2!\; a_2, \ldots, (n-k+1)!\; a_{n-k+1}), \tag{4}$$

Participating the partial Bell polynomials [7, 10-12], with the recurrence relation [3, 6, 8, 13]:

$$\sum_{k=0}^{n} a_k\; p(n-k) = 0, \tag{5}$$

Discovered by MacMahon [2, 3, 13].

On the other hand, from [14, 15] we have that relations type (1) are equivalent to the following Hessenberg determinant:

$$p(n) = (-1)^n \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & \cdots & a_1 \end{vmatrix}, \tag{6}$$

Obtained by Malenfant [13]. We note that the definition (2) gives the values:

Correponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México. E-mail: jlopezb@ipn.mx.

$$a_j = \begin{cases} 1, & j = 0, 5, 7, 22, 26, 51, 57, 92, 100, 145, 155, \dots \\ -1, & j = 1, 2, 12, 15, 35, 40, 70, 77, 117, 126, 176, \dots \\ 0 & otherwise \end{cases} \tag{7}$$

As examples of similar expressions to (1), (4), (5) and (6), we know the results [5]:

$$(x;q)_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_q q^{\binom{k}{2}} x^k, \qquad \frac{1}{(x\,;q)_{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k}_q x^k, \tag{8}$$

Then it is immediate the relation:

$$\sum_{k=0}^{\infty} \frac{1}{(q\,;q)_k} x^k = \frac{1}{\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q\,;q)_k} x^k}, \qquad (q;q)_0 = 1\,, \tag{9}$$

Therefore [7-9]:

$$\frac{1}{(q\,;q)_n} = \frac{1}{n!} \sum_{k=1}^{n} (-1)^k\, k!\; B_{n,k}\left(-\frac{1!\,q^{\binom{1}{2}}}{(q\,;q)_1}, \frac{2!\,q^{\binom{2}{2}}}{(q\,;q)_2}, -\frac{3!\,q^{\binom{3}{2}}}{(q\,;q)_3}, \frac{4!\,q^{\binom{4}{2}}}{(q\,;q)_4}, \dots, \frac{(-1)^{n-k+1}(n-k+1)!\,q^{\binom{n-k+1}{2}}}{(q\,;q)_{n-k+1}}\right), \tag{10}$$

Involving the partial Bell polynomials, with the recurrence property:

$$\sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{(q\,;q)_k\,(q\,;q)_{n-k}} = 0, \qquad n \geq 1. \tag{11}$$

We can apply to (10) the Birmajer-Gil-Weiner's inversion process [16] to obtain:

$$\frac{(-1)^n q^{\binom{n}{2}}}{(q\,;q)_n} = \frac{1}{n!} \sum_{k=1}^{n} (-1)^k\, k!\; B_{n,k}\left(\frac{1!}{(q\,;q)_1}, \frac{2!}{(q\,;q)_2}, \frac{3!}{(q\,;q)_3}, \frac{4!}{(q\,;q)_4}, \dots, \frac{(n-k+1)!}{(q\,;q)_{n-k+1}}\right). \tag{12}$$

Besides, from [14, 15] we have that relations type (9) are equivalent to the following Hessenberg determinant:

$$\frac{q^{\binom{n}{2}}}{(q\,;q)_n} = \begin{vmatrix} \frac{1}{(q;q)_1} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & 0 & \cdots & 0 \\ \frac{1}{(q;q)_3} & \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 1 \\ \frac{1}{(q;q)_n} & \frac{1}{(q;q)_{n-1}} & \frac{1}{(q;q)_{n-2}} & \cdots & \cdots & \frac{1}{(q;q)_1} \end{vmatrix}. \tag{13}$$

Similarly, Jha [17] deduced the following connection between the sum of divisors function and the numbers (7):

$$(n-1)!\;\sigma(n) = \sum_{k=1}^{n} (-1)^k\,(k-1)!\; B_{n,k}(1!\;a_1, 2!\;a_2, \dots, (n-k+1)!\;a_{n-k+1}), \tag{14}$$

Therefore its inverse is given by:

$$n!\;a_n = \sum_{k=1}^{n} (-1)^k\; B_{n,k}\big(0!\;\sigma(1), 1!\;\sigma(2), 2!\;\sigma(3), \dots, (n-k)!\;\sigma(n-k+1)\big). \tag{15}$$

The r – coloured partition function $p_r(n)$ is defined in terms of the Euler function [18, 19]:

$$[E(q)]^r = \sum_{n=0}^{\infty} p_r(n)\, q^n\,, \tag{16}$$

For example, if $r = 0, 1$ then (3) and (16) imply the values:

$$p_0(n) = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases}, \qquad p_1(n) = a_n \, . \tag{17}$$

From (1) and (16):

$$\sum_{n=0}^{\infty} p_r(n)\, q^n = \frac{1}{\left[\sum_{k=0}^{\infty} p(k)\, q^k\right]^r}, \tag{18}$$

Thus [15]:

$$\begin{vmatrix} rp(1) & 1 & 0 \ \cdots \ \cdots \ \cdots & 0 \\ 2rp(2) & (r+1)p(1) & 2 \ \cdots \ \cdots \ \cdots & 0 \\ 3rp(3) & (2r+1)p(2) & (r+2)p(1) \ \cdots \ \cdots \ \cdots & 0 \\ \vdots & (3r+1)p(3) & (2r+2)p(2) \ \cdots \ \cdots \ \cdots & \vdots \\ \vdots & \vdots & (3r+2)p(3) \ \cdots \ \cdots \ \cdots & \vdots \\ \vdots & \vdots & \vdots \ \cdots \ \cdots \ \cdots & \vdots \\ nrp(n) & \big((n-1)r+1\big)p(n-1) & \big((n-2)r+2\big)p(n-2) \ \cdots \ \cdots \ \cdots & (r+n-1)p(1) \end{vmatrix} =$$

$$= (-1)^n\, n!\; p_r(n), \qquad n \geq 1, \tag{19}$$

With its inverse [10, 17]:

$$p(n) = \sum_{r=0}^{n} (-1)^r \binom{n+1}{r+1} p_r(n), \qquad n \geq 0. \tag{20}$$

For example, if $n = 1,2,3,4$ then (19) implies the relations:

$$p_r(1) = -r,\ p_r(2) = \frac{r}{2!}(r-3),\ p_r(3) = -\frac{r}{3!}(r-1)(r-8),\ p_r(4) = \frac{r}{4!}(r-1)(r-3)(r-14), \tag{21}$$

Johnson [5] exhibits the following connection between the Euler and divisors functions:

$$\sum_{n=1}^{\infty} \sigma(n)\, q^n = -q\, \frac{d}{dq} \log\ (q;q)_\infty = -\frac{q}{E(q)}\, \frac{d}{dq}\, E(q) \overset{(3)}{=} -\frac{1}{E(q)}\, (a_1 q + 2a_2 q^2 + 3a_3 q^3 + \cdots),$$

$$\overset{(1)}{=} -\left[\sum_{k=0}^{\infty} p(k)\, q^k\right]\left[\sum_{m=0}^{\infty} m\ a_m\, q^m\right] \overset{[20]}{=} -\sum_{n=1}^{\infty} q^n \left[\sum_{j=0}^{n} p(j)\ (n-j)\ a_{n-j}\right],$$

Where we can apply (5) to obtain the formula:

$$\sigma(n) = \sum_{k=1}^{n} k\ a_{n-k}\, p(k) = -\sum_{k=1}^{n} k\ a_k\, p(n-k), \tag{22}$$

Deduced by Osler-Hassen-Chandrupatla [3]; with (5) and (22) it is immediate the known expression:

$$p(n) = \frac{1}{n} \sum_{j=0}^{n-1} p(j)\, \sigma(n-j), \qquad n \geq 1; \tag{23}$$

Finally, we indicate the interesting recurrence relation [3]:

$$\sigma(n) = -n\, a_n - \sum_{k=1}^{n-1} a_{n-k}\ \sigma(k), \qquad n \geq 2, \tag{24}$$

And the identity [17]:

$$B_{n,k}(1!\ a_1, 2!\ a_2, \ldots, (n-k+1)!\ a_{n-k+1}) = \frac{n!}{k!} \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r}\ p_r(n). \tag{25}$$

It is simple to show that (14) and (25) imply the relationship [17]:

$$\sigma(n) = n \sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r} p_r(n), \qquad n \geq 1. \tag{26}$$

Remark . - From (19) for $r = 9,\ n = 4$:

$$p_9(4) = \frac{1}{4!} \begin{vmatrix} 9 & 1 & 0 & 0 \\ 36 & 10 & 2 & 0 \\ 81 & 38 & 11 & 3 \\ 180 & 84 & 40 & 12 \end{vmatrix} = -90 \equiv 0 \quad mod\ 5,$$

In agreement with the congruence $p_9(5m+4) \equiv 0\ mod\ 5$ indicated by Forbes [18]; besides, from (19) with $r = -1,\ n = 5$:

$$p_{-1}(5) = \frac{1}{5!} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 4 & 0 & 2 & 0 & 0 \\ 9 & -2 & 1 & 3 & 0 \\ 20 & -6 & 0 & 2 & 4 \\ 35 & -15 & -3 & 2 & 3 \end{vmatrix} = 7 = p(5),$$

Which it is correct because $p_{-1}(n) = p(n)$. We note that $p_{24}(n-1)$ is the Ramanujan's τ-function.

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