Studies in Nonlinear Sciences 4 (3): 32-33, 2019
ISSN 2221-3910

DOI: 10.5829/idosi.sns.2019.32.33

Some Properties of Legendre Polynomials

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Abstract: Rangarajan-Purushothaman's formula for the generalized derivative allows deduce some properties of Legendre polynomials.

Key words: Legendre polynomials - Generalized derivative

INTRODUCTION

Rangarajan-Purushothaman [1, 2] obtained the following generalization of the Lanczos derivative [3, 4]:

$$f^{(m)}(x) = lim_{\varepsilon\to 0} \frac{(2m+1)!!}{2\,\varepsilon^{m+1}} \int_{-\varepsilon}^{\varepsilon} P_m\left(\frac{t}{\varepsilon}\right) f(x+t)\,dt, m = 1, 2, \ldots \quad (1)$$

Involving the Legendre polynomials [5-7].

If $f(x) = 1$, then (1) implies the property:

$$\int_0^1 P_n(u)\,du = 0, n = 2, 4, 6, \ldots \quad (2)$$

From (1) for $f(x) = x^N$:

$$\int_{-1}^1 P_n(u)\,u^k\,du = 0, k < n, \quad (3)$$

$$\int_0^1 P_n(u)\,u^n\,du = \frac{n!}{(2n+1)!!} = \frac{2^n\,(n!)^2}{(2n+1)!}, n = 0, 1, 2, \ldots \quad (4)$$

On the other hand, we know the relations:

$$\int_0^1 P_{2l}(u)\,u^m\,du = \frac{(-1)^l\,\Gamma\left(l-\frac{m}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{2\,\Gamma\left(-\frac{m}{2}\right)\Gamma\left(l+\frac{m+3}{2}\right)}, m > -1, \quad (5)$$

$$\int_0^1 P_{2l+1}(u)\,u^m\,du = \frac{(-1)^l\,\Gamma\left(l+\frac{1-m}{2}\right)\Gamma\left(1+\frac{m}{2}\right)}{2\,\Gamma\left(l+2+\frac{m}{2}\right)\Gamma\left(\frac{1-m}{2}\right)}, m > -2, \quad (6)$$

Thus (4) can be deduced from (5) and (6) for $m = n = 2l$ and $m = n = 2l + 1$, respectively.

We have the following Schmied's formula (2005) [8]:

$$u^m = \sum_{l=m,m-2,\ldots} \frac{m!\,(2l+1)}{2^{\frac{m-l}{2}}\left(\frac{m-l}{2}\right)!\,(m+l+1)!!} P_l(u), \quad (7)$$

Therefore [9]:

$$\int_{-1}^1 P_n(u)\,u^m\,du = \frac{2^{n+1}\binom{\frac{m+n}{2}}{n}}{(m+1)\binom{m+n+1}{n}}, m - n = 0, 2, 4, \ldots, \quad (8)$$

which for $m = n$ implies (4).

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The Legendre polynomials can be written in terms of the Gauss hypergeometric function:

$$P_n(x) = \frac{(2n-1)!!}{n!} \sum_{k=0}^{n} \binom{n}{k} \; {}_2F_1(k-n, -n;\; -2n;\; 2)\, x^k, \tag{9}$$

and we know the result:

$${}_2F_1(-n, -n;\; -2n; 2) = \begin{cases} 0, n = 1, 3, 5, \ldots \\ \frac{(-1)^{\frac{n}{2}}\, n!\, (n-1)!!}{n!!\, (2n-1)!!}, n = 2, 4, 6, \ldots \end{cases}, \tag{10}$$

Then from (9) and (10):

$$P_n(0) = {}_2F_1\left(-n, n+1; 1; \frac{1}{2}\right) = \begin{cases} 0, n = 1, 3, 5, \ldots \\ \frac{(-1)^{\frac{n}{2}}\, (n-1)!!}{n!!}, n = 2, 4, 6, \ldots \end{cases}. \tag{11}$$

Finally, the expression:

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k\, (1-x)^k\, (1+x)^{n-k} \binom{n}{k}^2, \tag{12}$$

and (11) imply the relation:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0, n = 1, 3, 5, \ldots \\ \frac{(-1)^{\frac{n}{2}}\, 2^n\, (n-1)!!}{n!!}, n = 2, 4, 6, \ldots \end{cases}. \tag{13}$$

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