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Accelerated Solution of High Order Non-linear ODEs using Chebyshev Spectral Method Comparing with Adomian Decomposition Method

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Abstract: In this article, an accurate Chebyshev spectral method for solving high order non-linear ODEs is presented. Properties of the Chebyshev polynomials are utilized to reduce the computation of the problem to a set of algebraic equations. Some examples are given to verify and illustrate the efficiency and simplicity of the method. We compared our numerical results against the Adomian decomposition method (ADM). Special attention is given to study the convergence analysis of ADM. Numerical results were obtained from these two methods show that the proposed techniques are in excellent conformance in most cases. Also, from the presented examples, we found that the proposed method can be applied to wide class of high order non-linear ODEs.

Key words: Chebyshev spectral method • high order non-linear ODEs • Gauss-Lobatto nodes • Adomian decomposition method • convergence analysis

INTRODUCTION

Chebyshev polynomials are examples of eigenfunctions of singular Sturm-Liouville problems. Chebyshev polynomials have been used widely in the numerical solutions of the boundary value problems [5] and in computational fluid dynamics. The existence of a fast Fourier transform for Chebyshev polynomials to efficiently compute matrix-vector products has meant that they have been more widely used than other sets of orthogonal polynomials. Chebyshev polynomials are well known family of orthogonal polynomials on the interval [-1,1] that have many applications [10, 19]. They are widely used because of their good properties in the approximation of functions. However, with our best knowledge, very little work was done to adapt this poly-nomials to the solution of differential equations. The well known Chebyshev polynomials [7] are defined on the interval [-1,1] and can be determined with the aid of the following recurrence formula:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n = 1,2,\ldots$$

The first three Chebyshev polynomials are

$$T_0(x) = 1, \; T_1(x) = x, \; T_1(x) = 2x^2-1$$

The square integrable function f(x) can be expressed in Chebyshev series:

$$f_m(x) \cong \sum_{k=0}^{m} c_k T_k(x)$$

where:

$$c_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{f(0.5x+0.5)T_0(x)}{\sqrt{1-x^2}}dx$$

$$c_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(0.5x+0.5)T_k(x)}{\sqrt{1-x^2}}dx, \; k = 1,2,\ldots,m$$

This technique has been employed to solve a large variety of linear and non-linear differential equations. This method is used for solving second and fourth-order elliptic equations [13]. This method is also adopted for solving fractional order integro-differential equations [19]. Also, this procedure is used to obtain the numerical solution of ODEs with non-analytic solution [6].

Ordinary differential equations have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, visco-elasticity, biology, physics and engineering [6, 8, 9]. Consequently, considerable attention has been given to the solutions of high-order ODEs of physical interest. Most differential equations do not have exact solutions, so approximation and numerical techniques [14-20], must be used. Recently, several numerical methods to solve the differential equations have been given such as variational iteration method [11, 18, 20, 23], homotopy perturbation method [18, 21], Adomian decomposition

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method [1-4, 8, 12, 22-26] and collocation method [13, 14].

In this paper, differential equations with a more complex nonlinearity are considered. We consider the numerical solution of high-order ODEs which containing the polynomial functions of unknown and its derivatives and can be written in the form:

$$\sum_{k=0}^{m}\sum_{r=0}^{n} P_{k,r}(x)u^{r}(x)u^{(k)}(x) + \sum_{k=1}^{m}\sum_{r=1}^{m} Q_{k,r}(x)u^{(r)}(x)u^{(k)}(x) = f(x) \tag{1}$$

subject to the following initial conditions:

$$u^{(i)}(0) = c_i, \qquad i = 0,1,2,...,m \tag{2}$$

where

$$u^{(0)}(x) = u(x), \; u^{0}(x) = 1$$

and u(x) is an unknown function from $C^m[0,a]$. Known functions $P_{k,r}(x)$, $Q_{k,r}(x)$ and f(x) are defined on the closed interval [0,a]. This is a non-linear differential equation of order m. The first and the second order problems are often encountered in the literature. Important non-linear equations of this type are Riccati, Abel, Emden-Fowler, Duffing, Van der Pol, Rayleigh and Yermakovs equations. These equations arise in different areas of physics and engineering sciences.

The organization of this paper is as follows: In the next section, a brief review of the Adomian decomposition method is introduced. Section 3 summarizes the convergence analysis of ADM. In section 4, some illustrative examples are given and solved using the two methods, Chebyshev spectral method and ADM to clarify the methods. Also a conclusion is given in section 5.

Note that we have computed the numerical results using Mathematica programming.

A BRIEF REVIEW OF THE ADOMIAN DECOMPOSITION METHOD

Consider the non-linear initial value problem of ODEs in the following general form:

$$L^{s}u(x) = Ru(x) + N(x(x)) + g(x) \tag{3}$$

with the initial conditions:

$$u^{(k)}(0) = f_k, \; k = 0,1,2,\ldots,s-1 \tag{4}$$

for some constants f_k, where

$$L^{s} = \frac{d^{s}}{dx^{s}}, \quad s = 1,2,3,...$$

(the highest derivative with respect to x) and R are linear operators, i.e., it is possible to find numbers $n_1>0$, $n_2>0$ such that $\|L^{S}u\| \le n_1\|u\|, \|Ru\| \le n_2\|u\|$. The non-linear term N(u) is Lipschitz continuous with:

$$|N(u) - N(\theta)| \le r|u-\theta|, \qquad \forall\; x \in J = [0,a]$$

for any constant r > 0.

Apply the inverse operator L^{-S} which defined by:

$$L^{-S}(\,.\,) = \int_0^{(x_s)}\int_0^{(x_{s-1})} \cdots \int_0^{(x_1)} (\,.\,)\,(dt_1)\ldots(dt_{s-1})(dt_s) \tag{5}$$

to both sides of (3) gives:

$$u(x) = \sum_{i=0}^{s-1} \frac{f_i}{i!}x^{i} + L^{-s}[Ru(x) + N(u(x)) + g(x)] \tag{6}$$

where the first part from the right hand side of Eq. (6) is obtained from the solution of the homogenous differential equation $L^{S}u(x) = 0$ using the initial conditions.

Adomian decomposition method [3, 4] defines the solution u(x) as an infinite series in the form:

$$u(x) = \sum_{i=0}^{\infty} u_i(x) \tag{7}$$

where the components $u_i(x)$ can be obtained in recursive form. The non-linear term N(u) can be decomposed by an infinite series of polynomials given by:

$$N(u) = \sum_{i=0}^{\infty} A_i \tag{8}$$

where the components A_i can be obtained using the following formula:

$$A_k = \frac{1}{k!}\left[\frac{d^{k}}{d\lambda^{k}} N\left(\sum_{i=0}^{\infty}\lambda^{i}u_i\right)\right]_{\lambda=0}, \qquad k = 0,1,2,... \tag{9}$$

Substitute by (7) and (8) into Eq.(6) gives:

$$\sum_{m=0}^{\infty} u_m(x) = \sum_{i=0}^{s-1} \frac{f_i}{i!} x^i + L^{-s}[g(x)] + L^{-s}[R \sum_{m=0}^{\infty} u_m(x) + \sum_{m=0}^{\infty} A_m] \quad (10)$$

From this equation, we can obtain the components $u_m(x)$ of the solution u(x) by the following recurrence formula:

$$u_0(x) = \sum_{i=0}^{s-1} \frac{f_i}{i!} x^i + L^{-s}[g(x)]$$

$$u_{m+1}(x) = L^{-s}[Ru_m(x) + A_m], \quad m \geq 0 \quad (11)$$

The convergence of ADM is introduced in many papers, for example [12, 17].

CONVERGENCE ANALYSIS OF ADM

In this section, the sufficient conditions are presented to guarantee the convergence of ADM, when applied to solve non-linear ODEs, where the main point is that we prove the convergence of the series, which is generated by using ADM.

Theorem 1: The non-linear problem (3) has a unique solution, whenever $0<\alpha<1$, where,

$$a = (n_1 + n_2)\frac{a^s}{s!}$$

where the constants n_1 and n_2 are defined above.

Proof: Denoting (C[J],||.||) the Banach space of all continuous functions on J with the norm defined by:

$$||f(x)|| = \max_{x \in J} |f(x)|$$

Define a mapping $F: E \to E$ where

$$Fu(x) = s(x) + L^{-s}[Ru(x) + N(x(x)) + g(x)]$$

where

$$\sigma(x) = \sum_{i=0}^{s-1} \frac{f_i}{i!} x^i$$

is the solution of the homogeneous differential equation $L^S u(x) = 0$ using the initial conditions.

Now, let us assume that, u and u^* be two different solutions to (3), then, we can obtain:

$$|Fu - Fu^*| = \max_{x \in J} |L^{-s}[R(u - u^*) + N(u) - N(u^*)]|$$

$$\leq \max_{x \in J} [\ |R(u - u^*)| + |N(u) - N(u^*)|\ ] \int_0^x \ldots s-\text{fold}-\ldots \int_0^x dx$$

$$\leq \max_{x \in J} (n_1 |u - u^*| + n_2 |u - u^*|) \int_0^x \ldots s-\text{fold}-\ldots \int_0^x dx$$

$$\leq (n_1 + n_2) \frac{a^s}{s!} \max_{x \in J} |u - u^*| = a||u - u^*||$$

Under the condition $0<\alpha<1$, the mapping F is contraction, therefore, by the Banach fixed-point theorem for contraction, there exist a unique solution to problem (3) and this completes the proof.

Hosseini and Nasabzadeh introduced a simple method to determine the rate of convergence of Adomian decomposition method [12]. In this section, we adapt it to seek here.

Theorem 2: Let N be an operator from a Hilbert space into itself and u(x) be the exact solution of Eq.(3), then, the approximate solution which is obtained by (11) converges to u(x) if: $0<\gamma<1$ and satisfy the following condition:

$$||u_{k+1}|| \leq \gamma ||u_k||, \quad k = 0,1,2,... \quad (12)$$

Proof: The proof of this theorem can be found in [12].

ILLUSTRATIVE EXAMPLES

In this section, we introduce three examples of non-linear ODEs. We find the numerical solution of these examples using the Chebyshev spectral method and ADM and plot the curves of these solutions.

Example 1: Let us consider Van der Pol equation [9]:

$$u^{(2)}(x) - 0.05(1 - u^2(x))\, u'(x) + u(x) = 0, \ 0 < x < a \quad (13)$$

with the conditions $u(0) = 0$, $u'(0) = 0.5$.

1.I: Procedure solution using Chebyshev spectral method: We solve the non-linear ODEs of the form (13) with the given initial conditions by using Chebyshev spectral method. For this purpose since the Gauss-Lobatto nodes lie in the computational interval [-1,1], in the first step of this method, the transformation $x = \frac{a}{2}(\eta + 1)$ is used to change Eq.(13) to the following form:

$$u^{(2)}(\eta) - 0.05\left(\frac{a}{2}\right)(1-u^2(\eta))u'(\eta) + \left(\frac{a}{2}\right)^2 u(\eta) = 0, \; -1<\eta<1 \tag{14}$$

The transformed initial conditions are given by:

$$u(-1) = 0, \quad u'(-1) = 0.5\left(\frac{a}{2}\right) \tag{15}$$

where

$$u^{(0)}(\eta) = u(\eta), \quad u^{0}(\eta) = 1$$

and $u(\eta)$ is the unknown function from $C^m[-1,1]$. Where the differentiation in Eqs.(14)-(15) will be with respect to the new variable η. Our technique is accomplished by starting with a Chebyshev approximation for the highest order derivative, $u^{(2)}(\eta)$ and generating approximations to the lower order derivatives $u^{(i)}$, i = 0,1, as follows: Setting, $u^{(2)}(\eta) = \varphi(\eta)$, then by integration we obtain:

$$\begin{aligned} u^{(1)}(\eta) &= \int_{-1}^{\eta} \varphi(\eta)d\eta + c_0 \\ u(\eta) &= \int_{-1}^{\eta}\int_{-1}^{\eta} \varphi(\eta)\, d\eta d\eta + (\eta+1)\, c_0 + c_1 \end{aligned} \tag{16}$$

From the initial conditions (15), we can obtain the constants of integration c_i, i = 0,1, where $c_0 = \frac{a}{4}, c_1 = 0$

Therefore, we can give approximations to Eq.(14) as follows:

$$u_i = \sum_{j=0}^{n} \ell_{ij}^{u}\varphi_j + d_i^{u}, \quad u_i^{(1)} = \sum_{j=0}^{n} \ell_{ij}^{u1}\varphi_j + d_i^{u1} \tag{17}$$

for all i = 0,1,2,…,n, where

$$\ell_{ij}^{u} = b_{ij}^{2}, \; \ell_{ij}^{u1} = b_{ij}, \; d_i^{u} = (\eta_i + 1)c_0 + c_1, \; d_i^{u1} = c_0$$

where

$$b_{ij}^{2} = (\eta_i - \eta_j)b_{ij}$$

and b_{ij} are the elements of the matrix B as given in Ref. [10]. By using Eq. (17), one can transform Eq. (14) to the following system of non-linear equations in the highest derivative:

$$\varphi_i - 0.05\left(\frac{a}{2}\right)\left(1-\left(\sum_{j=0}^{n}\ell_{ij}^{u}\varphi_j + d_i^{u}\right)^2\right)\left(\sum_{j=0}^{n}\ell_{ij}^{u1}\varphi_j + d_i^{u1}\right) + \left(\frac{a}{2}\right)^2\left(\sum_{j=0}^{n}\ell_{ij}^{u}\varphi_j + d_i^{u}\right) = 0 \tag{18}$$

This scheme is a non-linear system of n+1 algebraic equations in n+1 unknowns φ_i, which then solved using Newton's iteration method. After solving this system and substitute φ_i in Eq.(17), we can obtain the numerical solution of Eq.(13).

The convergence of this method is given in [10].

1.II: Procedure solution using ADM: In order to obtain the numerical solutions of Eq.(13) using ADM, we follow the following steps:

1: Eq.(13) can be rewritten in the operator form:

$$L^2u(x) = Ru(x) + N(u(x)) \tag{19}$$

where

$$L^2 = \frac{d^2}{dx^2}, \quad Ru = -u(x), \quad Nu(x) = 0.05(1-u^2(x))\, u'(x)$$

2: Apply the inverse operator L^{-2} which defined by (5) to both sides of Eq.(19) gives:

$$u(x) = \sum_{k=0}^{1} \frac{x^k}{k!}u^{(k)}(0) - L^{-2}[u(x)] + L^{-2}[Nu(x)] \tag{20}$$

Substituting by Eqs.(7) and (8) in Eq.(20) gives:

$$\sum_{m=0}^{\infty} u_m(x) = \sum_{k=0}^{1} \frac{x^k}{k!}u^{(k)}(0) - L^{-2}\left[\sum_{m=0}^{\infty} u_m(x)\right] + L^{-2}\left[\sum_{m=0}^{\infty} A_m\right] \tag{21}$$

substituting by initial conditions, then, the components $u_m(x)$ of the solution u(x) can be written as:

$$u_0(x) = 0.5x, \; u_{m+1}(x) = -L^{-2}[u_m(x)] + L^{-2}[A_m], \; m \ge 0 \tag{22}$$

where A_m can be obtained by the formula (9).

Table 1: The Chebyshev solution, u_{Ch} and the solution using ADM, u_{ADM}

X	u_{ADM}	u_{Ch}
0	0	0
0.0669873	0.033525	0.033525
0.25	0.124476	0.124476
0.5	0.242704	0.242704
0.75	0.34715	0.34715
0.933013	0.410924	0.410924
1	0.431051	0.431051

Table 2: The convergence behavior of the truncated solutions using ADM

Method	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$
ADM	0.234545	0.154723	0.023578

3: The components $u_m(x)$ of the solution u(x) using the iteration formula (22) are given as follows:

$$u_0(x) = 0.5x$$
$$u_1(x) = 0.0125x^2 - 0.0833333x^3 - 0.000520833x^4$$
$$u_2(x) = 0.000208333x^3 - 0.00208333x^4 + 0.00413021x^5 + 0.00019097x^6$$

So, the solution u(x) can be approximated as:

$$u(x) \cong \psi_m(x) = \sum_{i=0}^{m} u_i(x) \qquad (23)$$

The truncated solution $\psi_2(x)$ is given by

$$\psi_2(x) = u_0(x) + u_1(x) + u_2(x)$$

The behavior of the numerical solutions using Chebyshev spectral method, u_{Ch}, with n = 12, compared with the approximate solution using ADM, u_{ADM}, with three components (m = 3) are presented in Table 1.

The convergence analysis of the approximate solution using ADM is given in Table 2, in terms of Theorem 1 such that the L_2-norm which is defined as

$$\|u(x)\| = \int_0^a |u(x)|^2 dx$$

Example 2: Consider the non-linear initial value problem:

$$u^{(3)}(x) + 16\,u(x) + 30e^{2x}u^3(x) = 45\,e^{-x},\; 0 < x < a \qquad (24)$$

subject to the following initial values

$$u(0) = 1,\quad u'(0) = -1,\quad u''(0) = 1$$

The exact solution of this problem is given by $u(x) = e^{-x}$.

2.I: Procedure solution using Chebyshev spectral method: We solve the non-linear ODEs of the form (24) with initial conditions by using Chebyshev spectral method. For this purpose since the Gauss-Lobatto nodes lie in the computational interval [-1,1], in the first step of this method, the transformation $x = \frac{a}{2}(\eta + 1)$ is used to change Eq.(24) to the following form:

$$u^{(3)}(\eta) + 16\left(\frac{a}{2}\right)^3 u(\eta) + 30\left(\frac{a}{2}\right)^3 e^{2\eta}u^3(\eta) = 45\left(\frac{a}{2}\right)^3 e^{-\eta},\; -1 < \eta < 1 \qquad (25)$$

The transformed initial conditions are given by:

$$u(-1) = 1,\quad u^{(1)}(-1) = -\left(\frac{a}{2}\right),\quad u^{(2)}(-1) = \left(\frac{a}{2}\right)^2 \qquad (26)$$

where $u^{(0)}(\eta) = u(\eta), u^0(\eta) = 1$ and $u(\eta)$ is the unknown function from $C^m[-1,1]$. Where the differentiation in Eqs.(25)-(26) will be with respect to the new variable η. Our technique is accomplished by starting with a Chebyshev approximation for the highest order derivative, $u^{(3)}$ and generating approximations to the lower order derivatives $u^{(i)}$, i = 0,1,2, as follows:

Setting $u^{(3)}(\eta) = \varphi(\eta)$, then by integration we obtain $u^{(2)}(\eta), u^{(1)}(\eta)$ and $u(\eta)$ as follows:

$$u^{(2)}(\eta) = \int_{-1}^{\eta} \varphi(\eta)d\eta + c_0$$
$$u^{(1)}(\eta) = \int_{-1}^{\eta}\int_{-1}^{\eta} \varphi(\eta)\, d\eta d\eta + (\eta + 1)\, c_0 + c_1 \qquad (27)$$
$$u(\eta) = \int_{-1}^{\eta}\int_{-1}^{\eta}\int_{-1}^{\eta} \varphi(\eta)\, d\eta d\eta d\eta + \frac{(\eta+1)^2}{2!}c_0 + \frac{(\eta+1)}{1!}c_1 + c_2$$

From the initial conditions (26), we can obtain the constants of integration u_i, i = 0,1,2 where

$$c_0 = \left(\frac{a}{2}\right)^2,\quad c_1 = -\left(\frac{a}{2}\right),\quad c_2 = 1$$

Therefore, we can give approximations to Eq. (24) as follows:

$$u_i = \sum_{j=0}^{n} \ell_{ij}^{u}\varphi_j + d_i^{u},\ u_i^{(1)} = \sum_{j=0}^{n} \ell_{ij}^{u1}\varphi_j + d_i^{u1},\ u_i^{(2)} = \sum_{j=0}^{n} \ell_{ij}^{u2}\varphi_j + d_i^{u2} \tag{28}$$

for all i = 0,1,2,…,n, where

$$\ell_{ij}^{u} = b_{ij}^{3},\ \ell_{ij}^{u1} = b_{ij}^{2},\ \ell_{ij}^{u2} = b_{ij},\ d_i^{u} = \sum_{i=0}^{2}\frac{(\eta_i+1)^i}{i!}c_{2-i},\ d_i^{u1} = (\eta_i+1)c_0 + c_1,\ d_i^{u2} = c_0$$

where

$$b_{ij}^{2} = (\eta_i - \eta_j)b_{ij},\qquad b_{ij}^{3} = \frac{(\eta_i-\eta_j)^2}{2!}b_{ij}$$

and b_{ij} are the elements of the matrix B as given in Ref. [10]. By using Eq.(28), one can transform Eq.(25) to the following system of non-linear equations in the highest derivative:

$$\varphi_i + 16\left(\frac{a}{2}\right)^3\left(\sum_{j=0}^{n}\ell_{ij}^{u}\varphi_j + d_i^{u}\right) + 30\left(\frac{a}{2}\right)^3 e^{2\eta_i}\left(\sum_{j=0}^{n}\ell_{ij}^{u}\varphi_j + d_i^{u}\right)^3 = 45\left(\frac{a}{2}\right)^3 e^{-\eta_i} \tag{29}$$

This scheme is a non-linear system of n+1 algebraic equations in n+1 unknowns φ_i, which then solved using Newton's iteration method. After solving this system and substitute φ_i in Eq.(28), we can obtain the numerical solution of Eq.(24).

2.II: Procedure solution using ADM: In order to obtain the numerical solutions for Eq.(24) using ADM, we follow the following steps:

1: Eq.(24) can be rewritten in the operator form:

$$L^3u(x) = Ru(x) + N(u(x)) + g(x) \tag{30}$$

where

$$L^3 = \frac{d^3}{dx^3},\ Ru = -16u(x),\ Nu(x) = -30e^{2x}u^3(x),\ g(x) = 45\,e^{-x}$$

First, to overcome the complicated excitation from the functions e^{-x} and e^{2x}, which can cause difficult integrations and proliferation of terms, we use the Taylor expansion of the functions at x = 0, in the following form:

$$e^{-x} \approx 1 - x + 0.5x^2,\ e^{2x} \approx 1 + 2x + 2x^2$$

2: Apply the inverse operator L^{-3} which defined by (5) to both sides of Eq.(30) gives:

$$u(x) = \sum_{k=0}^{2}\frac{x^k}{k!}u^{(k)}(0) + L^{-3}[45(1-x+0.5x^2)] + L^{-3}[Ru(x)] + L^{-3}[Nu(x)] \tag{31}$$

Substituting by Eqs.(7) and (8) in Eq.(31) gives:

$$\sum_{m=0}^{\infty} u_m(x) = \sum_{k=0}^{2} \frac{x^k u^{(k)}(0)}{k!} + L^{-3}[45(1-x+0.5x^2)]-16L^{-3}\left[\sum_{m=0}^{\infty} u_m(x)\right] + L^{-3}\left[\sum_{m=0}^{\infty} A_m\right] \quad (32)$$

substituting by initial conditions, then, the components $u_m(x)$ of the solution u(x) can be written as:

$$u_0(x) = 0.5x^2 - x + 1 + L^{-3}[45(1-x+0.5x^2)], \quad u_{m+1}(x) = -16L^{-3}\left[u_m(x)+A_m\right], \quad m \geq 0 \quad (33)$$

where A_m can be obtained by the formula (9).

3: The components $u_m(x)$ of the solution u(x) using the iteration formula (33) are given as follows:

$$u_0(x) = 1 - x + 0.5x^2$$

$$u_1(x) = -0.17x^3 + 0.04x^4 - 0.38x^5 + 0.25x^6 - 0.46x^7 + 0.38x^8 - 0.19x^9 + 0.05x^{10} - 0.01x^{11}$$

$$u_2(x) = 7.5x^3 - 1.875x^4 + 0.147x^6 - 0.021x^7 + 0.121x^8 - 0.082x^9 + 0.120x^{10} + ... - 0.0006x^{19}$$

Table 3: The exact solution, the Chebyshev solution, u_{Ch} and the solution using ADM, u_{ADM}

x	u_{ADM}	u_{Ch}	u_{ex}
0	1	1	1
0.0669873	0.935207	0.935207	0.935207
0.25	0.728801	0.728801	0.728801
0.5	0.626531	0.626531	0.626531
0.75	0.312367	0.312367	0.312367
0.933013	0.53367	0.53367	0.53367
1	0.547879	0.547879	0.547879

Table 4: The convergence behavior of the truncated solutions using ADM

Method	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$
ADM	0.243101	0.697275	0.052006

So, the solution u(x) can be approximated as:

$$u(x) \cong \psi_m(x) = \sum_{i=0}^{m} u_i(x) \quad (34)$$

The truncated solution $\psi_2(x)$ is given by

$$\psi_2(x) = u_0(x) + u_1(x) + u_2(x)$$

The behavior of the numerical solutions using Chebyshev spectral method, u_{Ch}, with n = 12, compared with the approximate solution using ADM, u_{ADM}, with three components (m = 3) are presented in Table 3.

The convergence analysis of the approximate solution using ADM is given in Table 4, in terms of Theorem 1 such that the L_2-norm which is defined as:

$$\|u(x)\| = \int_0^a |u(x)|^2 dx$$

Example 3: Consider the non-linear initial value problem:

$$u^{(5)}(x) = e^{-x}u^2(x), \quad 0 < x < a \quad (35)$$

subject to the following initial values $u^{(i)}(0) = 1$, i = 0,1,2,3,4 and the exact solution of this problem is given by $u(x) = e^x$.

3.I: Procedure solution using Chebyshev spectral method: We solve the non-linear ODEs of the form (35) with initial conditions by using Chebyshev spectral method. For this purpose since the Gauss-Lobatto nodes lie in the computational interval [-1,1], in the first step of this method, the transformation $x = \frac{a}{2}(\eta + 1)$ is used to change Eq.(35) to the following form:

$$u^{(5)}(\eta) = \left(\frac{a}{2}\right)^5 e^{-\eta}u^2(\eta), \quad -1 < \eta < 1 \quad (36)$$

The transformed initial conditions are given by:

$$u^{(i)}(-1) = \left(\frac{a}{2}\right)^i, \; i = 0,1,2,3,4 \quad (37)$$

where

$$u^{(0)}(\eta) = u(\eta), \; u^0(\eta) = 1$$

and $u(\eta)$ is an unknown function from $C^m[-1,1]$. Where the differentiation in Eqs.(36)-(37) will be with respect to the new variable η. Our technique is accomplished by starting with a Chebyshev approximation for the highest order derivative, $u^{(5)}$ and generating approximations to the lower order derivatives $u^{(i)}$, i = 0,1,2,3,4 as follows:

Setting $u^{(5)}(\eta) = \varphi(\eta)$, then by integration we obtain $u^{(i)}(\eta)$, i = 0,1,2,3,4 as follows:

$$
\begin{aligned}
u^{(4)}(\eta) &= \int_{-1}^{\eta} \varphi(\eta)d\eta + c_0 \\
u^{(3)}(\eta) &= \int_{-1}^{\eta}\int_{-1}^{\eta} \varphi(\eta)\, d\eta d\eta + (\eta+1)\, c_0 + c_1 \\
u^{(2)}(\eta) &= \int_{-1}^{\eta}\int_{-1}^{\eta}\int_{-1}^{\eta} \varphi(\eta)\, d\eta d\eta d\eta + \sum_{i=0}^{2} \frac{(\eta+1)^i}{i!} c_{2-i} \\
u^{(1)}(\eta) &= \int_{-1}^{\eta}\int_{-1}^{\eta}\int_{-1}^{\eta}\int_{-1}^{\eta} \varphi(\eta)\, d\eta d\eta d\eta d\eta + \sum_{i=0}^{3} \frac{(\eta+1)^i}{i!} c_{3-i} \\
u(\eta) &= \int_{-1}^{\eta}\int_{-1}^{\eta}\int_{-1}^{\eta}\int_{-1}^{\eta}\int_{-1}^{\eta} \varphi(\eta)\, d\eta d\eta d\eta d\eta d\eta + \sum_{i=0}^{4} \frac{(\eta+1)^i}{i!} c_{4-i}
\end{aligned}
\tag{38}
$$

From the initial condiions (37), we can obtain the constants of integration c_i, i = 0,1,2,3,4 where $c_i = \left(\frac{a}{2}\right)^{4-i}$

Therefore, we can give approximations to Eq.(36) as follows:

$$
\begin{aligned}
&u_i = \sum_{j=0}^{n} \ell_{ij}^{u}\varphi_j + d_i^{u},\; u_i^{(1)} = \sum_{j=0}^{n} \ell_{ij}^{u1}\varphi_j + d_i^{u1},\; u_i^{(2)} = \sum_{j=0}^{n} \ell_{ij}^{u2}\varphi_j + d_i^{u2}, \\
&u_i^{(3)} = \sum_{j=0}^{n} \ell_{ij}^{u3}\varphi_j + d_i^{u3},\; u_i^{(4)} = \sum_{j=0}^{n} \ell_{ij}^{u4}\varphi_j + d_i^{u4}
\end{aligned}
\tag{39}
$$

for all i = 0,1,2,…,n, where

$$
\ell_{ij}^{u} = b_{ij}^{5},\; \ell_{ij}^{u1} = b_{ij}^{4},\; \ell_{ij}^{u2} = b_{ij}^{3},\; \ell_{ij}^{u3} = b_{ij}^{2},\; \ell_{ij}^{u4} = b_{ij},\; d_i^{u} = \sum_{i=0}^{4} \frac{(\eta_i+1)^i}{i!} c_{4-i},
$$

$$
d_i^{u1} = \sum_{i=0}^{3} \frac{(\eta_i+1)^i}{i!} c_{3-i},\; d_i^{u2} = \sum_{i=0}^{2} \frac{(\eta_i+1)^i}{i!} c_{2-i}\, d_i^{u3} = (\eta_i+1)c_0 + c_1,\; d_i^{u4} = c_0
$$

where

$$
b_{ij}^{2} = (\eta_i - \eta_j)b_{ij},\; b_{ij}^{3} = \frac{(\eta_i-\eta_j)^2}{2!} b_{ij},\; b_{ij}^{4} = \frac{(\eta_i-\eta_j)^3}{3!} b_{ij},\; b_{ij}^{5} = \frac{(\eta_i-\eta_j)^4}{4!} b_{ij}
$$

and b_{ij} are the elements of the matrix B as given in Ref. [10]. By using Eq.(39), one can transform Eq.(36) to the following system of non-linear equations in the highest derivative:

$$
\varphi_i - \left(\frac{a}{2}\right)^5 e^{-\eta_i} \left(\sum_{j=0}^{n} \ell_{ij}^{u}\varphi_j + d_i^{u} \right)^2 = 0 \tag{40}
$$

Table 5: The exact solution, the Chebyshev solution, u_{Ch} and the solution using ADM, u_{ADM}

x	u_{ADM}	u_{Ch}	u_{ex}
0	1	1	1
0.0669873	1.06928	1.06928	1.06928
0.25	1.28403	1.28403	1.28403
0.5	1.64872	1.64872	1.64872
0.75	2.117	2.117	2.117
0.933013	2.54216	2.54216	2.54216
1	2.71828	2.71828	2.71828

Table 6: The convergence behavior of the truncated solutions using ADM

Method	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$
ADM	0.145786	0.084504	0.012345

This scheme is a non-linear system of n+1 algebraic equations in n+1 unknowns φ_i, which then solved using Newton's iteration method. After solving this system and substitute φ_i in Eq.(39), we can obtain the numerical solution of Eq.(35).

3.II: Procedure solution using ADM: In order to obtain the numerical solutions for Eq. (35) using ADM, we follow the following steps:

1: Eq.(35) can be rewritten in the operator form:

$$L^5 u(x) = N(u(x)) \tag{41}$$

where

$$L^5 = \frac{d^5}{dx^5}, \; Nu(x) = e^{-x} u^2(x)$$

First, to overcome the complicated excitation from the functions e^{-x}, which can cause difficult integrations and proliferation of terms, we use the Taylor expansion of the fnctions at x = 0, in the following form $e^{-x} \approx 1 - x + 0.5x^2$.

2: Apply the inverse operator L^{-5} which defined by (5) to both sides of Eq. (41) gives:

$$u(x) = \sum_{k=0}^{4} \frac{x^k}{k!} u^{(k)}(0) + L^{-5}[Nu(x)] \tag{42}$$

Substituting by Eqs.(7) and (8) in Eq.(42) gives:

$$\sum_{m=0}^{\infty} u_m(x) = \sum_{k=0}^{4} \frac{x^k u^{(k)}(0)}{k!} + L^{-5}\left[\sum_{m=0}^{\infty} A_m\right] \tag{43}$$

substituting by initial conditions, then, the components $u_m(x)$ of the solution u(x) can be written as:

$$\begin{aligned} u_0(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \\ u_{m+1}(x) &= L^{-5}\left[A_m\right], \; m \geq 0 \end{aligned} \tag{44}$$

where A_m can be obtained by the formula (9).

3: The components $u_m(x)$ of the solution u(x) using the iteration formula (44) are given as follows:

$$u_0(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

$$u_1(x) = \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^8}{10080} - \frac{x^9}{22680} - \frac{x^{10}}{72576} - \frac{13x^{11}}{3991680} - \frac{x^{12}}{1710720} - \frac{7x^{13}}{88957440} - \frac{x^{14}}{138378240}$$

$$u_2(x) = \frac{x^{10}}{1814400} + \frac{x^{11}}{19958400} - \frac{x^{12}}{11404800} - \frac{x^{13}}{21621600} + \ldots + \frac{5x^{23}}{2413820423798784} + \frac{x^{24}}{8469545346662400}$$

So, the solution u(x) can be approximated as:

$$u(x) \cong \psi_m(x) = \sum_{i=0}^{m} u_i(x) \tag{45}$$

The truncated solution $\psi_2(x)$ is given by

$$\psi_2(x) = u_0(x) + u_1(x) + u_2(x)$$

The behavior of the numerical solutions using Chebyshev spectral method, u_{Ch}, with n = 12, compared with the approximate solution using ADM, u_{ADM}, with three components (m = 3) are presented in Table 5.

The convergence analysis of the approximate solution using ADM is given in Table 6, in terms of Theorem 1 such that the L_2-norm which is defined as

$$||u(x)|| = \int_0^a |u(x)|^2 dx$$

CONCLUSION AND REMARKS

In this paper, the Chebyshev spectral method and the Adomian decomposition method are implemented to obtain the numerical solutions of the high-order non-linear ODEs and the numerical solutions with respect to these methods are compared. It is generally very difficult to find the analytical solutions of higher-order non-linear ODEs with variable coefficients. So, we interest in this article with using the two proposed methods to solve numerically such these equations. Since, we know that the Chebyshev polynomial approximation method is valid in the interval [-1,1], so, we used the transformation $x = \frac{a}{2}(\eta + 1)$ to change the interval [0,a]. The Chebyshev spectral method reduces the considered non-linear differential equation to a non-linear system of algebraic equations, which solved using the well known method, namely, Newton iteration method.

Also, by using ADM the solutions may take the closed form of the exact solution. In general since the ADM solves the problems on a few steps later of iteration satisfying the desired precision, it does not need more calculation in order to solve the differential equation. Special attention is given to study the convergence of ADM and satisfy this theoretical study in view the introduced numerical examples. In the end, from our numerical results using two proposed methods we can conclude that, the solutions are in excellent agreement with the exact solution in most cases. Also, the obtained results demonstrate reliability and efficiency of the proposed methods.

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