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An Approximation of the Analytic Solution of the Helmholtz Equation

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Abstract: In this article we discuss the analytic solution of the linear Helmholtz partial differential equation. The analytic solution is obtained by using differential transform method. Some numerical examples are presented to illustrate the efficiency and reliability of the method.

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INTRODUCTION

Consider the Helmholtz equation

$$\Delta^2 u + f(x,y)u = g(x,y) \qquad (1.1)$$

with the initial and boundary conditions:

$$u(0,y) = \psi_1(y), \quad u_x(0,y) = \psi_2(y) \qquad (1.2)$$

$$u(x,0) = \psi_3(x), u_y(x,0) = \psi_4(x) \qquad (1.3)$$

where ψ_1 (y), ψ_2 (y), ψ_3 (x) and ψ_4 (x) are given functions. These equations appear in such diverse phenomena as: elastic waves in solid including vibrating string, bars, membranes, sound or acoustics, electromagnetic waves and nuclear reactors [1, 2].

Several techniques, such as the Adomian's decomposition method and the finite difference method [3] and the variational iteration method [4] have been used to handle the Helmholtz equation numerically and analytically. In this paper, we implement the differential transform method (DTM) for finding analytic approximate solutions of the Helmholtz equation (1.1) and then a numerical comparison with the exact solution is demonstrated through three different examples.

The differential transform method was first applied by Zhou [5] who solved linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method computationally takes long time for large orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solution of ordinary or partial differential equations. The present method is well addressed in [6-8].

DEFINITIONS AND OPERATIONS OF THE TWO-DIMENSIONAL DIFFERENTIAL TRANSFORM

Consider a function of two variables u(x,y) and suppose that it can be represented as a product of two single-variable functions, i.e., u(x,y) = f (x) g(y). Based on the properties of one-dimensional differential transform, u(x,y) can be written as [9]:

$$u(x,y) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)y^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} U(i,j)x^i y^j \qquad (2.1)$$

where U(i,j) = F(i) G(j) is called the spectrum of u(x,y).

The basic definitions and operations of two-dimensional differential transform were introduced in [7, 9, 10] as follows:

Definition 1: If the function u(x,y) is analytic and continuously differentiable with respect to time x, y in the domain of interest, then let

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h} u(x,y)}{\partial x^k \partial^h y}\right]_{\substack{x=0\\y=0}} \qquad (2.2)$$

where the spectrum U(k,h) is the transformed function, which is also called T-function in brief. In this paper,

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Table 1: Operations of the two-dimensional differential transform

Original function	Transformed function
$u(x,y)=u_1(x,y)\pm u_2(x,y)$	$U(k,h)=U_1(k,h)\pm U_2(k,h)$
$u(x,y)=\alpha u(x,y)$	$U(k,h)=\alpha U_1(k,h)$
$u(x,y)=\frac{\partial u_1(x,y)}{\partial x}$	$U(k,h)=(k+1)U_1(k+1,h)$
$u(x,y)=\frac{\partial u_1(x,y)}{\partial y}$	$U(k,h)=(h+1)U_1(k,h+1)$
$u(x,y)=\frac{\partial^{r+s}u_1(x,y)}{\partial x^r\partial y^s}$	$U(k,h)=(k+1)(k+2)\ldots(k+r)(h+1)(h+2)\ldots\times(h+s)U_1(k+r,h+s)$
$u(x,y)=u_1(x,y)u_2(x,y)$	$U(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U_1(r,h-s)\ U_2(k-r,s)$
$u(x,y)=x^m y^m$	$U(k,h)=\delta(k-m,h-n)=\delta(k-m)\delta(h-n)=\begin{cases}1 & \text{for } k=m \text{ and } h=n,\\ 0 & \text{otherwise.}\end{cases}$

Table 2: Different approximate solutions and absolute errors for example 3.1

x	y	Exact solution	DTM	ADM [3]	Absolute errors
-1	-1.0	-1.910960076	-1.909747299	-1.909747299	0.001212777
-0.8	-0.8	-1.429411128	-1.491500900	-1.491500900	0.000261038
-0.6	-0.6	-1.040566113	-1.040530331	-1.040530331	0.000035782
-0.4	-0.4	-0.700556967	-0.700554815	-0.700554815	0.000002152
-0.2	-0.2	-0.367759501	-0.367759484	-0.367759484	0.000000017
0.0	0.0	0.000000000	0.000000000	0.000000000	0.000000000
0.2	0.2	0.448293902	0.448293884	0.448293884	0.000000018
0.4	0.4	1.029158828	1.029156415	1.029156415	0.000002413
0.6	0.6	1.804550411	1.804507931	1.804507931	0.000042480
0.8	0.8	2.850380700	2.850052490	2.850052490	0.000328210
1.0	1.0	4.261362463	4.259747299	4.259747299	0.001615164

the lowercase u(x,y) represents the original function while the uppercase U(k,h) stands for the transformed function (T-function).

Definition 2: The differential inverse transform of U(k,h) is defined as

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k y^h \qquad (2.3)$$

Combining Eqs. (2.2) and (2.3), it can be obtained that

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial y^h}u(x,y)\right]_{\substack{x=0\\y=0}}x^k y^h \qquad (2.4)$$

By using (2.2)-(2.4), the fundamental mathematical operations performed by the two-dimensional differential transform can readily be obtained and are listed in Table 1.

APPLICATIONS AND NUMERICAL RESULTS

We shall illustrate the numerical scheme by three examples. These examples are somewhat artificial in the sense that the exact answer is known in advance and the initial and boundary conditions are directly taken from this answer. Nonetheless, such an approach is needed to evaluate the accuracy of the numerical scheme. All the results are calculated by using the symbolic calculus software Mathematica.

Example 3.1: Consider a special case of the Helmholtz equation:

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}-u(x,y)=0 \qquad (3.1)$$

with the initial conditions

$$u(0,y)=y, \qquad u_x(0,y)=y+\cosh(y) \qquad (3.2)$$

Taking the differential transform of Eq.(3.1), then

$$\begin{aligned} U(k,h) &= (k+1)(k+2)U(k+2,h) \\ &\quad + (h+1)(h+2)U(k,h+2) \end{aligned} \tag{3.3}$$

From the initial conditions (3.2), all spectra can be found that

$$U(k,h) = \begin{cases} \dfrac{1}{k!h!} & \text{for } k=1 \text{ and } h \text{ is even, or } h=1 \text{ and } k \text{ is arbitrary} \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

Therefore, the closed form for the solution can be easily written as

$$\begin{aligned} u(x,y) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k y^h \\ &= y\sum_{k=0}^{\infty}\frac{x^k}{k!} + x\sum_{h=0}^{\infty}\frac{y^{2h}}{(2h)!} = y\exp(x) + x\cosh(y) \end{aligned} \tag{3.5}$$

and our approximate solution is given by

$$\begin{aligned} u(x,y) &= y\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) \\ &\quad + x\left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^6}{6!} + \frac{y^8}{8!}\right) \end{aligned} \tag{3.6}$$

Table 2 shows the approximate solutions for equation (3.1) obtained by using the differential transform method and Adomian decomposition method [3]. It is clear that the approximations obtained using the two methods are in high agreement with those obtained using the exact solution. It is to be noted that six terms only were used in evaluating the approximate solution by using the decomposition series solution in [3].

Example 3.2: The second Helmholtz equation we consider is the following:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} + 8u(x,y) = 0 \tag{3.7}$$

with the initial conditions

$$u(0,y) = \sin(2y), \quad u_x(0,y) = 0 \tag{3.8}$$

The exact solution is $u(x, y)=\cos(2x)\sin(2y)$.
Taking the differential transform of Eq. (3.7), then

$$\begin{aligned} 8U(k,h) &= -(k+1)(k+2)U(k+2,h) \\ &\quad - (h+1)(h+2)U(k,h+2) \end{aligned} \tag{3.9}$$

From the initial condition (3.8), all spectra can be found that

$$U(k,h) = \begin{cases} (-1)^{\frac{k}{2}}(-1)^{\frac{h-1}{2}}\dfrac{2^k 2^h}{k!h!}, & k \text{ even and } h \text{ odd} \\ 0 & , \text{ otherwise} \end{cases}$$

Therefore, the series solution can be easily written as

$$\begin{aligned} u(x,y) &= \sum\sum U(k,h) \quad x^k \ y^h \\ &= \sum_{k \text{ even}}\sum_{h \text{ odd}}(-1)^{\frac{k}{2}}(-1)^{\frac{h-1}{2}}\frac{2^k 2^h}{k!h!}x^k y^h \end{aligned}$$

However, the closed form solution of Eq. (3.7) is

$$\begin{aligned} u(x,y) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(-1)^k(2x)^{2k}}{(2k)!}.\frac{(-1)^h(2y)^{2h+1}}{(2h+1)!} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k(2x)^{2k}}{(2k)!}\sum_{h=0}^{\infty}\frac{(-1)^h(2y)^{2h+1}}{(2h+1)!} \end{aligned}$$

or

$$u(x, y) = \text{cox}\,(2x)\sin(2y)$$

Table 3 displays the comparison of the differential transform method solution of Eq. (3.7) with the exact solution for some values of x and y.

Example 3.3: In this example we consider the following inhomogeneous Helmholtz equation: (note the exact solution is ($u(x, y)=xy$).

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} + u(x,y) = xy \tag{3.10}$$

with the initial conditions

$$u(0,y) = 0 \tag{3.11}$$

$$u_x(0,y) = y \tag{3.12}$$

Taking the differential transform of Eq. (3.10)

$$\begin{aligned} U(k,h) &= kh - (k+1)(k+2)U(k+2,h) \\ &\quad - (h+1)(h+2)U(k,h+2) \end{aligned} \tag{3.13}$$

From the initial conditions (3.11), we have

$$u(0,y) = 0 \Rightarrow \sum_{h=0}^{\infty} U(0,h)y^h = 0 \Rightarrow U(0,h) = 0 \text{ for all } h$$

From the initial conditions (3.12), we have

Table 3: Approximate solution and absolute errors for example 3.2

x	y	Exact solution	DTM	Absolute errors
0.0	0.0	0.000000000	0.000000	0.000000000
0.1	0.1	0.194709171	0.194533	1.765771×10^{-4}
0.2	0.2	0.358678045	0.358678	0.000000000
0.3	0.3	0.466019543	0.466020	0.000000000
0.4	0.4	0.499786801	0.499787	2.2579×10^{-8}
0.5	0.5	0.454648713	0.454649	2.43316×10^{-7}
0.6	0.6	0.33773159	0.337733	1.639955×10^{-6}
0.7	0.7	0.167494075	0.167502	7.909882×10^{-6}
0.8	0.8	-0.029187071	-0.029160	2.9580266×10^{-5}
0.9	0.9	-0.221260221	-0.221170	8.9927112×10^{-5}
1.0	1.0	-0.378401247	-0.378170	2.28184304×10^{-4}

Table 4: Different approximate solutions and absolute errors for example 3.3

x	y	Exact solution	DTM	Absolute errors
-1.0	-1.0	1.00	1.00	0
-0.8	-0.8	0.64	0.64	0
-0.6	-0.6	0.36	0.36	0
-0.4	-0.4	0.16	0.16	0
-0.2	-0.2	0.04	0.04	0
0.0	0.0	0.00	0.00	0
0.2	0.2	0.04	0.04	0
0.4	0.4	0.16	0.16	0
0.6	0.6	0.36	0.36	0
0.8	0.8	0.64	0.64	0
1.0	1.0	1.00	1.00	0

$$u_x(0,y) = y \Rightarrow \sum_{k=0}^{\infty} U(1,h)y^h = y \Rightarrow U(1,h) = \begin{cases} 1, h=1 \\ 0, \text{otherwise} \end{cases}$$

Now, the rest of the transformation coefficients can be evaluated from the (3.13) by substitution of the known values of the right-hand side and get them by simple manipulations. So we have the next result:

$$\begin{aligned}
&U(0,h) = 0, U(1,h) = \begin{cases} 1, h=1 \\ 0, o.w \end{cases}, U(k,1) = \begin{cases} 1, k=1 \\ 0, \text{otherwise}, \end{cases} \\
&U(2,h) = 0, U(k,0) = 0 \\
&U(3,h) = 0, U(k,2) = 0 \\
&U(4,h) = 0, U(k,3) = 0 \qquad (3.14) \\
&\quad \vdots \qquad\qquad \vdots \\
&\quad \vdots \qquad\qquad \vdots \\
&U(k,h) = \begin{cases} 1, k=1 \text{and} \quad h=1 \\ 0, \text{otherwise} \end{cases}
\end{aligned}$$

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k y^h = xy \qquad (3.15)$$

Table 4 exhibits the approximate solutions for Eq. (3.10) obtained using the differential transform method and the decomposition solution [3] for some values of x and y.

CONCLUSIONS

In this paper, the two-dimensional differential transform method has been successfully applied to linear Helmholtz partial differential equation. All the examples show that the results of the present method are in excellent agreement with the exact solution. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. Many of the results obtained in this paper can be extended to significantly more general classes of linear and nonlinear differential equations.

REFERENCES

1. Burden, R.L. and J.D. Faires, 1963. Numerical Analysis, PWS Publishing Company, Boston.
2. Gerald, C.F. and P.O. Wheatley, 1994. Applied Numerical Analysis, Addison Wesley, California.
3. El-Sayed, S.M. and D. Kaya, 2004. Comparing numerical methods for Helmholtz equation model problem. Applied Mathematics and Computation, 150: 763-773.
4. Momani, S. and S. Abuasad, 2004. Application of He’s variational iteration method to Helmholtz equation. Chaos, Solitons and Fractals, 15: 967-979.
5. Zhou, J.K., 1986. Differential Transformation and Its Applications for Electrical Circuits, Huazhong Univ. Press, Wuhan, China.
6. Chen, C.K. and S.H. Ho, 1999. Solving partial differential equations by two-dimensional transform method. Applied Mathematics and Computation, 106: 171-179.
7. Jang, M.J., C.K. Chen and Y.C. Liu, 2001. Two-dimensional differential transform for partial differential equations. Applied Mathematics and Computation, 121: 261-270.
8. Ayaz, F. and G. Oturanç, 2004. An approximate solution of Burger’s equation by differential transform method. Selçuk Journal of Applied Mathematics, 5: 15-24.
9. Bildik, N., A. Konuralp, O. Bek and S. Küçükarslan, 2006. Solution of different type of the partial differential equation by differential transform method and Adomian’s decomposition method. Applied Mathematics and Computation, 172: 551-567.
10. Kurnaz, A., G. Oturanç and M. Kiris, 2005. n-Dimensional differential transformation method for solving PDEs. International Journal of Computer Mathematics, 82: 369-380.