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What Is the Role of Visualization in Generalization Processes: The Case of Preservice Secondary Mathematics Teachers

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Abstract: In recent years there is a growing body of research which emphasizes the role of generalization in mathematics education. Some of this research investigates the role, the extent and constraints of visual thinking in the generalization processes via the use of metaphors. This study suggests that visualization has an essential role when making generalizations among relations and rules. Hence it is very important to encourage mathematics teachers to use visualizations in their teaching practices.

Key words: Mathematical generalization · Visualization

INTRODUCTION

Mathematical thinking has played an important role in the development of human civilization for over two millennia. Because of this, the nature of mathematical thinking; what it is, how it functions in the minds of mathematicians, has to be investigated. Mathematics deals with generalizations relating abstract ideas, so, the role of the generalization in the development of mathematical knowledge is very important in mathematics education. A significant aspect of mathematics is concerned with objectification and representing abstractions from reality. These representations generally appear visually or connect with visually sensed experiences. Visualization not only organizes data at hand in meaningful structures, but is also an important factor guiding the development of generalization. It's clear that visualization involves mental images that go through various transformations and thus aid the development of generalization.

The focus of this paper will be on the qualitative aspects of visualizations when secondary mathematics teachers discover generalizations. Interviews and observations during writings are used to describe, analyze and interpret the mathematical experiences of pre-service secondary school mathematics teachers that were able to formulate generalizations using visualizations.

Theoretical Framework

Generalization: Sriraman [1] defines generalization in mathematics education literature as the process by which one derives or induces from particular cases. Mitchelmore [2] has been able to group the meanings of generalization into three categories which he called G1, G2 and G3.

G1: A synonym for abstraction or concept: Davydov [3] defines it as 'finding and singling out properties in a whole class of similar objects' and Dreyfus [4] considered it to be the first stage of abstraction. Mitchelmore and White [5] use the same meaning with 'abstract-general' concepts.

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G2: An extension of an existing concept: These extensions have three aspects. First is empirical extension that occurs when a person finds other contexts to which an existing concept applies. Second is mathematical extension that occurs when one class of mathematical objects is embedded within a larger class based on a different similarity. Third is mathematical invention that occurs when a mathematician deliberately changes or omits one or more defining properties of a familiar concept to form a more general concept.

G3: A theorem relating existing concepts: Dubinsky [6] defines generalization as the process of by which 'an existing schema is represented and used in a new situation different from the previous one'. Also Skemp [7] wrote 'the process of mathematical generalization is a sophisticated and powerful activity. Sophisticated because it involves reflecting on the form of the method while temporarily ignoring its content. Powerful, because it makes conscious, controlled and accurate reconstructions of one's existing schemas- not only in the response to the demands of assimilation of the new situations as they are encountered but ahead of these demands' (p.58).

Sriraman [1] mentions Krutetskii's [8] claim that in order for students to correctly formulate generalizations, they had to abstract from specific content and single out similarities, the structures and relationships. He also writes Polya's [9] statements about the value of inducing generalizations by working through particular cases and creating links between patterns found in seemingly disparate situations or examples, which in turn leads to making an inductive leap and formulating the generality.

Visualization: Visualization is an important aspect of mathematical understanding, insight and reasoning. Mathematics is a subject that is concerned with objectification and representing abstractions from reality and many of these representations appear to be visual, having roots in visually sensed experiences [10]. Researchers have contributed different and useful ideas about visualization. The definition of visualization can be summed up by putting together the ideas from different authors. Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings [11-13].

Presmeg [14] states that visualization is an aid to understanding or a means towards an end and so one can therefore speak about visualizing a concept or a problem but not a diagram. Visualizing a concept or a problem refers to a mental image of the problem and to visualize a problem means to understand the problem in the terms of a diagram or visual image. Hence the visualization process is one of which involves visual imagery with or without a diagram, as an essential part of the method of solution [15]. Bishop [10] defines the ability for visual processing as follows; this ability involves visualization and the translation of the abstract relationships and non-figural information into visual representations and visual imagery. It is an ability or process that does not relate to the form of the stimulus material presented. Presmeg [15] for instance seems to locate visual imagery in the mind: she says a visual image is a mental scheme depicting visual or spatial information and this mental scheme can exist with or without the presence of the perceptual being visualized. Presmeg [14] lists five different kinds of imagery that she identified in her learners as concrete, pictorial imagery (pictures in the mind), pattern imagery (pure relationships depicted in a visual-spatial scheme), memory images of formula, kinesthetic imagery (involving muscular activity, e.g., fingers walking) and dynamic (moving) imagery.

Thornton [16], mentiones that visual thinking should be an integral part of students' mathematical experiences and discusses its importance in developing algebraic understanding, in proving a powerful problem-solving tool and in valuing a variety of learning styles. According to Arcavi [11], visualization is both the product and the process of creation, interpretation and reflection upon pictures and images. Fishbein [17] claimed that visualization not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution. Also, it is mentioned that visualization can be even more than that: it can be the analytical process itself which concludes with a solution which is general and formal. Arcavi quotes Goethe's in his paper: "We don't know what we see, we see what we know" (p.230). This explains many situations which students see during their generalizing process.

MATERIALS AND METHODS

For this study, the participants consisted of secondary mathematics teachers in one of Turkey's state university's education faculty. They entered an elective course and participated voluntary. Problems were given to the students and all work done by the students was non-collaborative. Students were encouraged to include all their scratch work. Each student was interviewed individually for approximately 30 min at the completion of each problem. During these interviews the students were asked to talk through the problem and their solution. The aim was to let students verbalize and visualize the thought processes they had employed while thinking about the problems during the discovery of generalizations. In order to exhibit the states of verbalization and visualization that lead to the discovery of generalizations, several case studies of students are presented. One of the sample problems that was assigned was as follows: *'How many straight lines can pass through n points whenever three of them are non collinear?'* This problem wasn't given to the students directly. At first two points, then three points and then four points... were given until the student recognize the generalization. It was observed during this processes how the students used visualization and verbalization. In all vignettes I=Authors, S= Student. In Figure 1 and Figure 2 there are visual representations of the students.

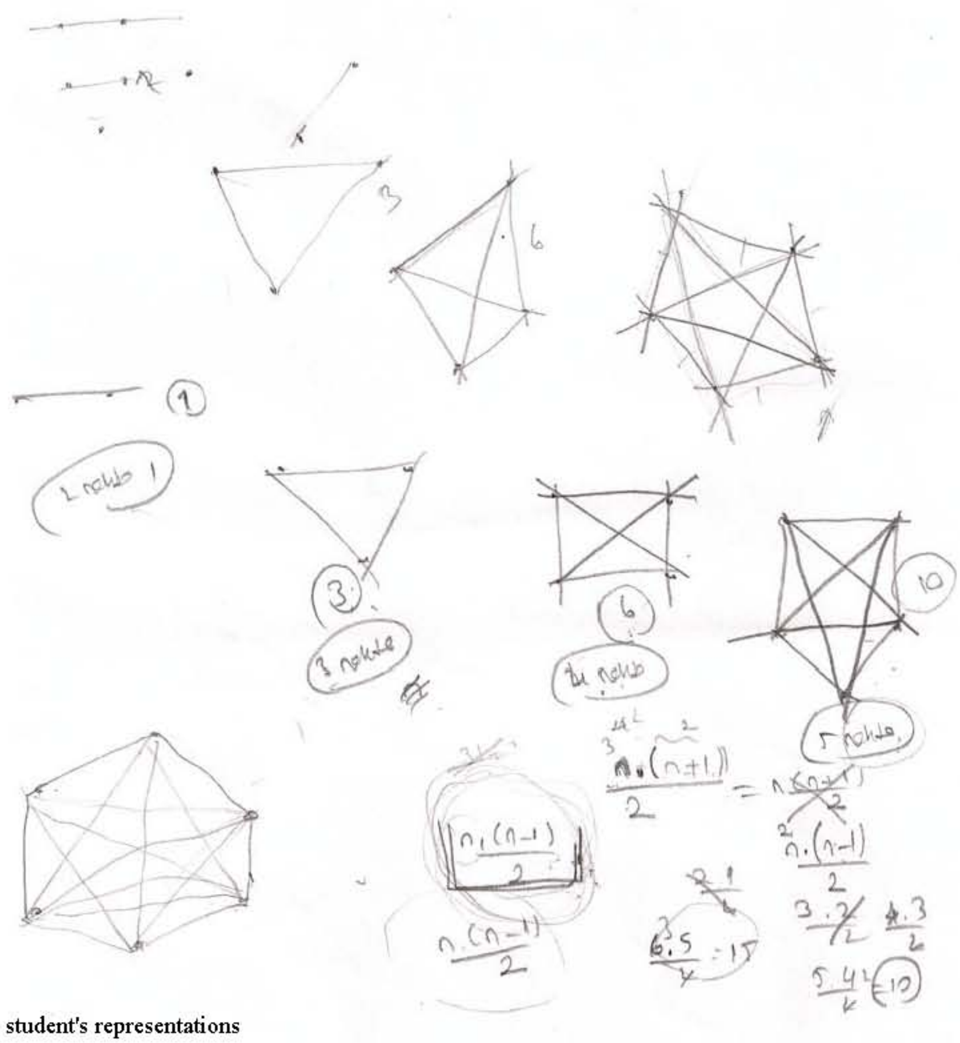

Fig. 1: First student's representations

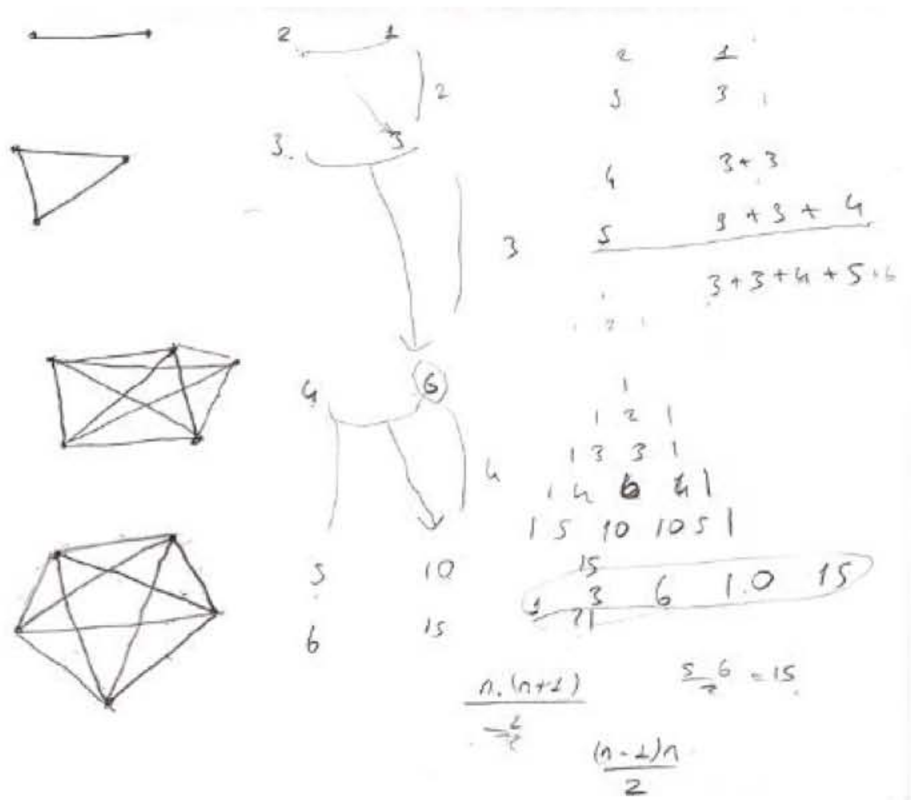

Fig. 2: Second student's representations

RESULTS AND DISCUSSION

First student was able to have the generalization and produced a formula. During generalization process, she used visualization. Visualization was important for her, because she used it in all steps. Figure 1 gives the visual representations and algebraically expressions.

Vignette 1:

I: If I give you two distinct points, how many straight lines do you pass from these points?

S: I can pass only one line (she draws two points and then pass a line from these points).

I: O.K. If I give you three distinct points that are nonlinear, how many straight lines do you pass from these points?

S: Hmm… (She draws three linear points and passes only one line from these points).I couldn't understand. The points are nonlinear? But two of them must be linear! Aren't they? (She thinks).O.K. (She draws three points like the corners of a triangle). I think it must be like a triangle.

I: O.K. If I give you four distinct points that three of them are nonlinear, how many straight lines do you pass from these points?

S: Hm…(She thinks). A line must pass from two points and so…. (she draws four distinct points that there of them are nonlinear and then passes the lines by counting). One, two, three, four, five and six. Hmm… six lines?

I: You have six lines. Do you?

S: Yes, I counted. There are six points.

I: O.K. Now, I give you five distinct points that three of them are nonlinear, how many straight lines do you pass from these points?

S: (She draws five points that three of them are nonlinear) I have five points. (she passes a line from two of them) One from here… (she continues to pass from the others one by one and counts) one from here, one from here, one from here… (she passes nine lines and counts every line). I have nine lines (she thinks). But it mustn't be nine? According to me I must have ten lines.

I: Why?

S: (she restart to count the lines that she drew).one, two, three,…, nine (she adds the last nine) and ten. I have ten lines.

I: Why did you think that you must have ten lines?

S: I thought to reduce to a formula.

I: O.K. Can you reduce a formula?

S: Hmm… Now … I passed one line from two points. (she draws again two points and a line and write 1 near the picture). Then I passed three lines from three points. (she draws again and write 3 near it). If I think four points (she Sdraws again four points and draws the lines by counting) one, two,…and six. O.K. I have six lines (and she writes 6 near the picture). So, If I think five points (she draws the same picture by counting) one... ten. I have ten lines. (she writes 10 near it and then she controls the pictures and thinks) one from two points, three from three points, six from four points, ten from five points. Hmm…I must formulate it with n. (she thinks) I think to have a sequence and then to reduce to a formula. (she writes the numbers of lines and thinks). I pass a line from two points (writes $N(N+1)/2$ and tries the numbers on this formula). Three lines from three points, so, I must write something in the parenthesis that must simplify the number 2, O.K. It must have 2 in it (she thinks and writes $N(N+1)/2$ and thinks the numbers) no, this isn't (she crosses out that she wrote and thinks). If I write $N(N+1)/2$?

I: Why did you do this?

S: When I try these p oints; one from two, three from three… (she tries the numbers of the points on the formula). The formula works for five points. If I try for six points (she writes for six points and has 15). I have 15 lines. ??? Is it true? Hmm… (she draws six points and passes lines from them by counting) one, two, …fourteen, fifteen. Yes, I have 15 lines. It is true.

I: Are you sure that it will work for all points?

S: Hmm… Why not? I found it for five points. Then I tried for six points and it worked. I saw. But anyway, I am not sure for all points….. (she thinks) But, I saw it works! It must be true!.

I: You are sure! O.K.

S: Yes.

Commentary on Case 1: In case 1 the student was sure that only one line would pass from two distinct points. But she still needed to draw two points and pass a line through them. This shows that visual thinking was important for her to be sure of her argument. Then when three distinct points that are noncollinear were given, she drew them linear and passed only one line from them. But during this time she recognized that three points were linear. And then she understood that two of these there points must be linear. Drawing picture of these points helped her decision. When four points were given she directly drew two points and added other two points that are non collinear and then drew the lines by counting. She passed six lines and she was sure about the points. Because she counted and couldn't find anymore points. So, visualization was her way to think about the possible lines. And when five points were given the same thing happened. But for five points she passed nine lines at first. Then she thought to find ten lines. During this time she decided to find a formula because the number of points and lines were increasing and she was getting confused about the number of the lines. So, she needed a formula. When reducing the formula, the denominator of the formula was 2. She recognized that a line passes through two points. So she put 2 in the formula. Then she thought $n(n+1)$ for the part of remain and she tried the numbers on this. The lines that she drew helped her to recognize that the formula was not true. Then she thought $n(n-1)$ and she tried again and found the formula. Also during this time visualization helped her to find the formula.

When the points were given, the student thought about the possible lines but then she needed a formula. This was the first step for her about the generalization process. Until the last step she used drawings and so visualization helped her during this process.

Case 2: The second student constructed a generalization and had a formula. He used visualization in generalization process. Figure 2 gives the visual representations and algebraically expressions.

Vignette 2:

I: If I give you two distinct points, how many straight lines do you pass from these points?

S: One line passes from two points (he draws two points and then passes a line from these points).

I: If I give you three distinct points that are nonlinear, how many straight lines do you pass from these points?

S: (he draws three points that are nonlinear and passes three lines from them).Three lines pass. One line from two points and three lines from three points (He writes 1 for 2 and 3 for 3 near their pictures.)

I: Now, if I give you four distinct points that three of them are nonlinear, how many straight lines do you pass from these points?

S: (He draws four distinct points that three of them are nonlinear and then passes the lines).Six points pass. (he writes 6 for 4 near its picture). O.K. Six lines passed from four points.

I: O.K. I give you five distinct points that three of them are nonlinear, how many straight lines do you pass from these points?

S: (he draws five points that three of them are nonlinear and passes the lines by counting) One, two, three,…ten (then he controls the lines by recounting silently) Hmm… (then he controls again silently). I have ten lines. Ten lines pass from five points.

I: You had ten lines. Have got more?

S: More? Immm (he controls the lines and points) No, only ten lines. I can't draw anymore. I went all points from this points, from this point, from this point. Yes I went all points from each point. (then he writes 10 for 5 near the picture.) I had ten lines for five points. (he looks the numbers that he wrote and thinks).

I: What do you think?

S: I think the numbers. Hmmm…

I: O.K. What can you say if I give you more distinct points that three of them are nonlinear? What do you think?

S: I think I must say it for n points (he thinks). I am looking to form a formula.

I: Can you have a formula?

S: I don't know. I must think. (he thinks on the numbers that he wrote near the pictures and writes the differences.) The difference between 1 and 3 is 2, the difference between 3 and 6 is 3 and the difference between 6 and 10 is 4. Hmm …(he thinks the sums of the points and the lines and writes the sums). Sum of 2 and 1 gives 3; 3 and 3 gives 6; 4 and 6 gives 10. I want to see the relation but (he thinks again and then writes the numbers by addition)? From 2 gives 1, from 3 gives 3, from 4 gives 3+3, from 5 gives 3+3+4 (he thinks) *n-1* is added for n.

It looks like Pascal Triangle. If I try for 6 points will I have 15 (he writes 15 for 6 near the picture)?

I: Will you have?

S: I think I will. (he adds one point to the picture for four points by drawing). When I add one point, one point is added for each one. When I add one point to 4 points, 4 lines are added. I mean that 4 lines are added to the number of lines for 5 points. So, I think, n-1 lines will be added to the number of line for n points. I must know $n\text{-}1^{th}$ to know n^{th}. So(he thinks)?

I: What do you see as a relation between the numbers?

S: (he writes the addition). If I say 3 plus 3 plus 4 plus 5 and then the next one will be 6. If I think Pascal triangle(he writes pascal triangle's numbers)... Now, there are 2, 3,6,10 and the next will have 15 and then 21… these are the numbers that I want. But how will I find the rule? (he thinks) O.K. I think the rule is $n(n+1)/2$. I must try on it. Hmm… For four points I have ten. For five points, I have fifteen. No! It doesn't work. I must change the rule (he thinks). O.K. The rule is $n(n-1)/2$. Because when I try, it works. Yes, For two points I have 1, for three points I have 3, for four points I have 6, for five points I have 10, for six points I have 15.Yes it's true.

Commentary on Case 2: In case 2, the student was able to say at once that a line passes through two distinct points. This means that he was sure but anyway he drew two points and passed the line. Here, visualization was his way to explain the right answer in his mind. And then when three nonlinear points were asked, he drew the picture at first and said three lines. Drawing the picture firstly was a way for his thinking process. He did the same things when four and five points were asked. When more lines were asked, he said 'I can't draw'. This answer shows that visualization (means drawing here) would give the right number for him. He wrote the numbers of the lines near their pictures. And then he started to think about these numbers. When the numbers for more points were asked, he wanted to have a rule. This was his first step for generalization. He thought about the relation between the points and the lines and during this time he drew a picture again. When he added the fifth point near the picture for four points, he recognized that n-1 lines would be added for n points. He was successful to generalize the relation between the numbers. Visual thinking was an essential way for him to find the relation and reducing the formula for constructing the generalization. He continued like that and at the last step, when he found the right answer, he still needed to try the numbers on the lines that he drew. And at last he was sure. Seeing the picture was his way to be sure. And also this shows the importance of visualization during the process.

CONCLUSIONS

Many students have difficulties to see the relations and discovery of generalization of the relations and so they hardly formulate the generalizations. Visualization has an importance in mathematical thinking and reasoning. In this study, it was investigated how the students use visualization during their thinking process to discover generalizations and create formulas of generalization. A simple problem that was given to the students was selected here to see this process. The reason was to see the process easily. We can understand from this research that students think their way through abstract situations via visualization and as Fischbein [17] said 'an essential factor for creating the feeling of self-evidence and immediacy' to construct the generalization and to have its formula. So, we can underline that visualization is an important process to discover the generalizations. And also it can be seen that 'visual forms of representation are important as legitimate elements of mathematical proofs' as Barwise and Etchemendy [18] claimed.

This study suggests that visualization has an essential role when making general relations and rules and therefore it is very important to encourage mathematics teachers to use visualizations in their teaching practices. According to Arcavi [11], seeing the thing itself, with the aid of technology which overcomes the limitation of our sight, provides not only a fulfillment of our desire to see and the subsequent enjoyment, but it may also sharpen our understanding, or serve as a springboard for questions which we were not able to formulate before. It seems that mathematics teachers can use visualization as pictures, images, diagrams on papers or with technological tools to have generalization process successfully on their students' mathematical thinking.

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