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A Mathematical Modeling of Prey-Predator Model with Holling-Type II and Modified Leslie-Gower Schemes with Prey Refuge

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Abstract: A predator-prey system with Holling type II functional response and modified Leslie–Gower type dynamics incorporating constant proportion of prey refuge compared by considering the model without prey refuge is considered. In both cases condition for local asymptotic stability of positive equilibrium point of the system is discussed by non-dimensionalize the system and global asymptotic stability is proved by defining appropriate Dulac function. Numerical simulations are also carried out to verify the analytical results.

Key words: Prey refuge · Local and Global stability · Limit cycle · modified Leslie-Gower

INTRODUCTION

The dynamic relationships between species and their complex properties are at the heart of many ecological and biological processes [1]. As was pointed out by [2], mite predator–prey interactions often exhibit spatial refugia which afford the prey some degree of protection from predation and reduce the chance of extinction due to predation. In [2], Tapan Kumar Kar had considered a predator–prey model with Holling type II response function and a prey refuge. The author obtained conditions on persistent criteria and stability of the equilibria and limit cycle for the system. For more works on this direction, one could refer to [2-5] and the references cited therein. Such system has been investigated by several researchers. In particular, the boundedness of solutions and global stability of the positive equilibrium points of the system has been studied by [6]. Sufficient conditions for the existence and global attractivity of positive periodic solutions of the model were discussed by [7].

Although many authors have considered the dynamic behaviors of the modified Leslie–Gower model [7-12] and predator–prey with a prey refuge [2-5,13], as far as we know, there are almost no literatures discussing the modified Leslie–Gower model with a prey refuge.

The Mathematical Model: The model considered is based on the assumption that a constant proportion $m \in (0,1)$ of the prey can take refuge to avoid predation, this leaves $(1-m)X$ of the prey available for predation. Let X (t) and Y (t) represent the population of the prey and predator species at any time t. The main feature of the model is that the interaction of species affects both populations. Terms representing logistic growth of the prey species in the absence of the predator are included in the prey equations. The model has two non-linear autonomous ordinary differential equations describing how the population densities of the two species would vary with time.

Thus, the model under the assumption with Holling type II functional response and the modified Leslie-Gower type predator dynamics is given by:-

$$\begin{cases} \dfrac{dX}{dT} = r\left(1-\dfrac{X}{K}\right)X - \dfrac{c_1(1-m)XY}{k_1+(1-m)X} \\ \dfrac{dY}{dT} = Y\left(s-\dfrac{c_2 Y}{k_2+(1-m)X}\right) \end{cases} \tag{1}$$

where all the parameters in the model assumes positive values and with initial value $X(0) \geq 0$ and $y(0) \geq 0$.

This two species food chain model describes a prey population x which serves as food for a predator y. The model parameters r, s, K, k_1, k_2, c_1 and c_2 are assuming only positive values. These parameters are defined as follows:

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r is per capita intrinsic growth rates for prey, s is gives the maximal per-capita growth rate of predator, K is the carrying capacity of the environment, k_1 (respectively, k_2) measures the extent to which environment provides protection to prey x (respectively, to the predator y), c_1 is the maximum value which per capita reduction rate of prey and c_2 is the crowding effect for the predator.

The following non-dimensional state variables and parameters are chosen.

$$x=\frac{X}{K},\ y=\frac{Y}{K},\ t=rT$$
$$a=\frac{c_1}{r},\ \beta=\frac{k_1}{K},\ \gamma=\frac{s}{r},\ \sigma=\frac{c_2}{r},\ \omega=\frac{k_2}{K}$$

The system (1) takes the following non-dimensional form

$$\begin{cases} \dfrac{dx}{dt}=(1-x)x-\left(\dfrac{a(1-m)xy}{\beta+(1-m)x}\right) \equiv F(x,y) \\ \dfrac{dy}{dt}=y\left(\gamma-\dfrac{\sigma y}{\omega+(1-m)x}\right) \equiv G(x,y) \end{cases} \quad (2)$$
$$x(0)=x_0\geq 0\ ;\ y(0)=y_0\geq 0$$

Lemma 2.1: All the solutions $(x(t),y(t))$ of the system (2) are nonnegative. That is $x(t)\geq 0$ $y(t)\geq 0$ for all $t\geq 0$.

Lemma 2.2: All the solution $(x(t),y(t))$ of the system (2) are bounded.

Proof: The first equation of (2) gives us

$$\frac{dx}{dt}=x\left(1-x-\frac{a(1-m)y}{\beta+(1-m)x}\right) < x(1-x)$$

Therefore, $\limsup_{t\to\infty} x(t)<1$. Hence, $x(t)$ is always bounded.

Similarly,

$$\frac{dy}{dt}=y\left(\gamma-\frac{\sigma y}{\omega+(1-m)x}\right) \leq y\left(\gamma-\frac{\sigma y}{\omega+1-m}\right) =\gamma y\left(1-\frac{y}{\omega+1-m}\frac{\sigma}{\gamma}\right) =\gamma y\left(1-\frac{y}{\omega+1-m}\lambda\right) =y\left(1-\frac{y}{(\omega+1-m)/\lambda}\right)$$

Therefore, we have $y(t)\leq \max\left\{\frac{\omega+1-m}{\lambda}, y(0)\right\}\equiv L$, $\lambda=\frac{\sigma}{\gamma}$.

Hence, the solutions $(x(t),y(t))$ of the system (2) with the given initial conditions are bounded.

Nonnegative Equilibria: Obviously, (2) has three boundary equilibria, E_0 (0, 0), E_1 (1, 0) and $E_2\left(0,\frac{\omega\gamma}{\sigma}\right)$.

Besides these equilibrium points the system (2) has one positive equilibrium points, say.

$E^*(x^*,y^*)$ is obtained by solving the following simultaneous equation

$$\frac{a(1-m)y^*}{\beta+(1-m)x^*}=1-x^*$$
$$y^*=\frac{\gamma(\omega+(1-m)x^*)}{\sigma}$$

One can easily see that x^* satisfies the quadratic equation

$$Ax^{*2}+Bx^*+C=0$$

Where,

$A=(1-m)\sigma, B=a\gamma m^2+(\sigma-2a\gamma)m+a\gamma+\sigma\beta-\sigma,\ C=a\gamma\omega-a\gamma\omega m-\sigma\beta.$

Stability Analysis

Local Stability: The local asymptotical stability of each equilibrium point is studied by computing the Jacobean matrix and finding the eigenvalues evaluated at each equilibrium point. For stability of the equilibrium points, the real parts of the eigenvalues of the Jacobean matrix must be negative.

Theorem 3.1.1: The trivial equilibrium E_0 is unstable.

Proof: At $E_0(0,0)$, the Jacobean matrix becomes

$$J(E_0)=\begin{pmatrix}1 & 0\\ 0 & \gamma\end{pmatrix}$$

Thus, the eigenvalues of this matrix are λ_1 = 1 and λ_2 = γ, both are positive, which shows that the trivial equilibrium is unstable.

Theorem 3.1.2: The equilibrium point $E_1(1,0)$ is also unstable.

Proof: The Jacobean matrix becomes

$$J(E_1)=\begin{bmatrix} -1 & \dfrac{-a(1-m)}{\beta+(1-m)} \\ 0 & \gamma \end{bmatrix}$$

The eigenvalues are λ_1 = -1<0, λ_2 = γ>0

Thus the equilibrium point E1(1,0) is unstable saddle point.

Theorem 3.1.3: The equilibrium point $E_2\left(0,\dfrac{\omega\gamma}{\sigma}\right)$ as locally asymptotically stable if $m<1-\dfrac{\sigma\beta}{\alpha\gamma\omega}$.

Proof: At $E_2\left(0,\dfrac{\omega\gamma}{\sigma}\right)$, the Jacobean matrix is

$$J(E_2)=\begin{pmatrix} 1-\dfrac{\alpha\gamma\omega(1-m)}{\sigma\beta} & 0 \\ \dfrac{\gamma^2}{\sigma} & -\gamma \end{pmatrix}$$

The eigenvalues of the matrix J(E_2) are $\lambda_1=1-\dfrac{\alpha\gamma\omega(1-m)}{\sigma\beta}$, $\lambda_2=-\gamma<0$

For E_2 to be locally asymptotically stable, we should have $\lambda_1<0$, This is true for $m<1-\dfrac{\sigma\beta}{\alpha\gamma\omega}$.

Theorem 4.1.4: The dynamic system (2) has $E^*(x^*,y^*)$ as locally asymptotically stable if $\gamma>\left(1-2x^*-\dfrac{\alpha\beta(1-m)y^*}{(\beta+(1-m)x^*)^2}\right)$.

Proof: At $E^*(x^*,y^*)$, the Jacobean matrix takes the form

$$J(E^*)=\begin{pmatrix} x^*\left(-1+\dfrac{\alpha(1-m)^2y^*}{(\beta+(1-m)x^*)^2}\right) & \dfrac{-\alpha(1-m)x^\circ}{\beta+(1-m)x^*} \\ \dfrac{(1-m)\gamma^2}{\sigma} & -\gamma \end{pmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$trace(J(E^*))=a_{11}+a_{22}=x^*\left(-1+\frac{\alpha(1-m)^2y^*}{(\beta+(1-m)x^*)^2}\right)-\gamma$$

Thus, $trace(J(E^*))<0$ if and only if $\gamma>x^*\left(-1+\dfrac{\alpha(1-m)^2y^*}{(\beta+(1-m)x^*)^2}\right)$.

$$\det(J(E^*))=a_{11}a_{22}-a_{12}a_{21}=\gamma x^*\left(\frac{(\beta+(1-m)x^*)^2\sigma+\alpha\gamma(1-m)^2(\beta-\omega)}{(\beta+(1-m)x^*)^2\sigma}\right)>0$$

Hence, the equilibrium point E^* is locally asymptotically stable provided $\gamma>-x^*-\dfrac{\alpha(1-m)^2x^*y^*}{(\beta+(1-m)x^*)^2}$.

Global Stability

Theorem 3.2.1: The system (2) does not admit any periodic solution for $m>1-\beta$.

Proof: Let $(x(t),y(t))$ be solutions of the system (2). Define Dulac function

$$H(x,y)=\frac{\beta+(1-m)x}{xy}$$

Then

$$Q=\frac{\partial(HF)}{\partial x}+\frac{\partial(HG)}{\partial y}$$
$$=-\left(\frac{(\beta-1+m)+2(1-m)x}{y}+\frac{(x(1-m)+\beta)\sigma}{x((1-m)x+\omega)}\right)$$

It is observed that Q<0 for m>1-β. Therefore, by Dulac criterion, the system (2) has no non-trivial periodic solutions.

Lemma 3.2.1: If m>1-β then the local asymptotical stability of the system (2) ensures its global asymptotical stability around the unique positive interior equilibrium point $E^*(x^*,y^*)$.

Section TWO

The Model Without Prey Refuge:

Consider when m = 0 that is, there is no prey refuge:

Here it is assumed that all the preys are accessible to the predator species, our mathematical model (1) becomes,

$$\begin{cases} \dfrac{dX}{dT}=r\left(1-\dfrac{X}{K}\right)X-\dfrac{c_1XY}{k_1+X} \\ \dfrac{dY}{dT}=Y\left(s-\dfrac{c_2Y}{k_2+X}\right) \end{cases} \tag{3}$$

where all the parameters in the model are positive.

The following non-dimensional state variables and parameters are chosen.

$$x=\frac{X}{k} \quad y=\frac{Y}{k} \quad t=rT$$
$$a=\frac{c_1}{r} \quad \beta=\frac{k_1}{K} \quad \gamma=\frac{s}{r} \quad s=\frac{c_2}{r} \quad \omega=\frac{k_2}{K}$$

The system (3) takes the following non-dimensional form

$$\begin{cases} \dfrac{dx}{dt}=\left(1-x-\dfrac{\alpha y}{\beta+x}\right)x \equiv F(x,y) \\ \dfrac{dy}{dt}=y\left(\gamma-\dfrac{\sigma y}{\omega+x}\right) \equiv G(x,y) \end{cases} \tag{4}$$

$x(0)=x_0{}^30;\ y(0)=y_0{}^30$

Equilibrium Points: We now study the existence of equilibria of system (4). All possible equilibria are

(I) The trivial equilibrium E_0 (0, 0)
(ii) Equilibrium in the absence of predator (y = 0) E_1 (1, 0)
(iii) Equilibrium in the absence of prey (x= 0) $E_2\left(0,\dfrac{\omega\gamma}{\sigma}\right)$
(iii) The interior (positive) equilibrium $E^*(x^*,y^*)$ where x^* is the unique positive root of the quadratic equation $\sigma x^{*2}+(\alpha\gamma+\sigma\beta-\sigma)x^*+\alpha\gamma\omega-\sigma\beta=0$;

$$x^*=\frac{-B+\sqrt{B^2-4\sigma C}}{2\sigma},\ y^*=\frac{\gamma(\omega+x^*)}{\sigma}$$

Where $B=\alpha\gamma+\sigma\beta-\sigma, C=\alpha\gamma\omega-\sigma\beta$

Theorem: The system (4) does not admit any periodic solution for $\beta>1$.

Proof: Let (x(t), y(t)) be solutions of the system (3.2). Define Dulac function

$$H(x,y)=\frac{\beta+x}{xy}$$

Then

$$Q=\frac{\partial(HF)}{\partial x}+\frac{\partial(HG)}{\partial y}$$
$$=-\left(\frac{(\beta-1)+2x}{y}+\frac{(x+\beta)\sigma}{x(x+\omega)}\right)$$

It is observed that $Q<0$ for $\beta>1$. Therefore, by Dulac criterion, the system (4) has no non-trivial periodic solutions.

Numerical Simulation: In this section we will solve the system equation (2) and (4) by using the in-built ordinary differential equation solver MatLab function ode45.

For solving system (2), we took the following parametric values. $\alpha=1,\ \gamma=0.2,\ \omega=0.2,\ \sigma=0.1,\ \beta=0.2$. For these values of parameter, we simplify the existence and stability properties of the equilibrium for the system.

For the given parametric values, it is found that the coexistence equilibrium point exists for $m>0.5$. Hence, in our simulation we took the values of m in the range $0.5>m>1$.

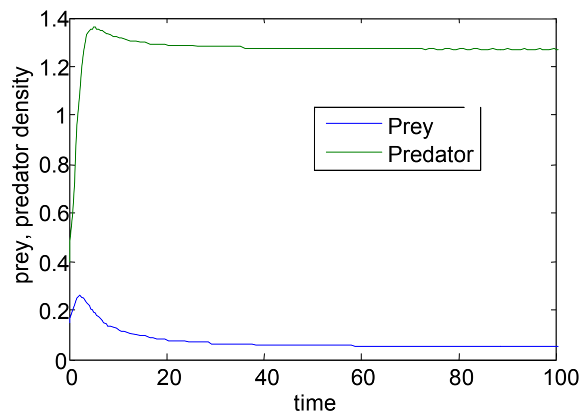

Fig. 1: Times series plot of prey and predator at m=0.55

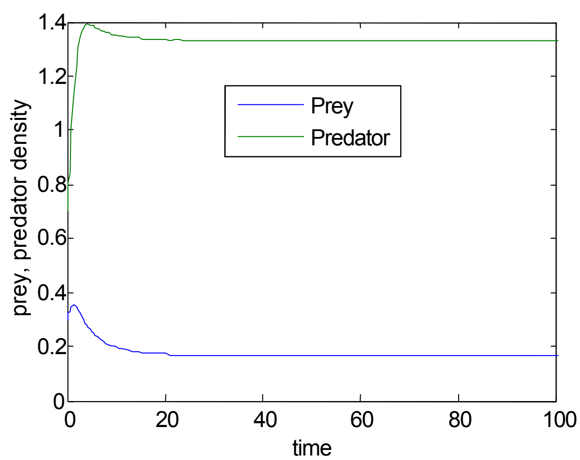

Fig. 2: Time series plot of prey and predator at m=0.6.

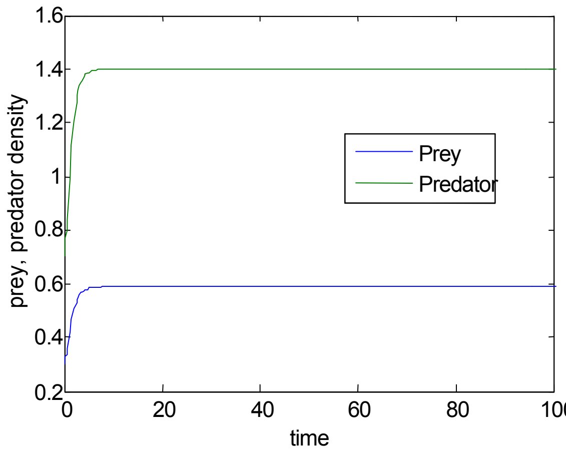

Fig. 3: Time series plot of prey and predator at m=0.8.

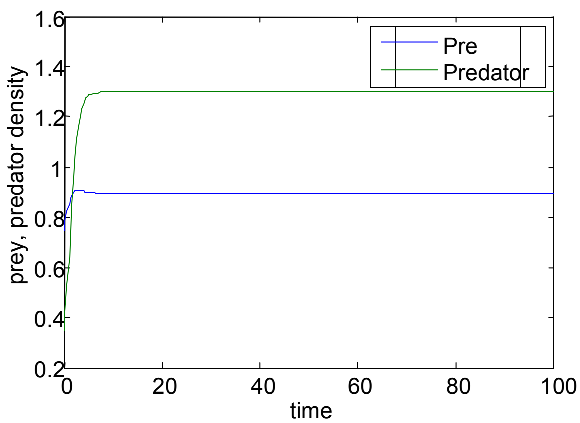

Fig. 4: Time series plot of prey and predator at m=0.95

For the system equation (4), that is the system in the absence of prey refuge, we have used the following parametric values as fixed and the parameter γ as a control parameter. These values are $\alpha = 1,\ \omega = 0.2, \sigma = 0.1, \beta = 0.2$. For these set of parametric values the coexistence equilibrium point exists whenever $\gamma<0.1$. The coexistence equilibrium point is locally asymptotically stable for $\gamma<0.651234$ and hence unstable otherwise.

Figures 5-7 shows the stability of the coexistence equilibrium point. That is; the solution, trajectory, of the prey and predator species approaches to the coexistence equilibrium point.

A figure 8 shows the existence of a limit cycle, periodic solution. Figure 9 also shows the oscillatory nature of the predator prey system. Figure 10 represents the instability of the coexistence equilibrium point.

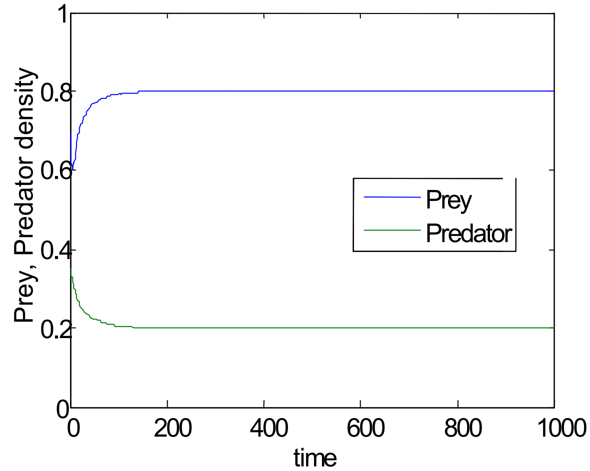

Fig. 5: series plot of the prey and predator at γ=0.02

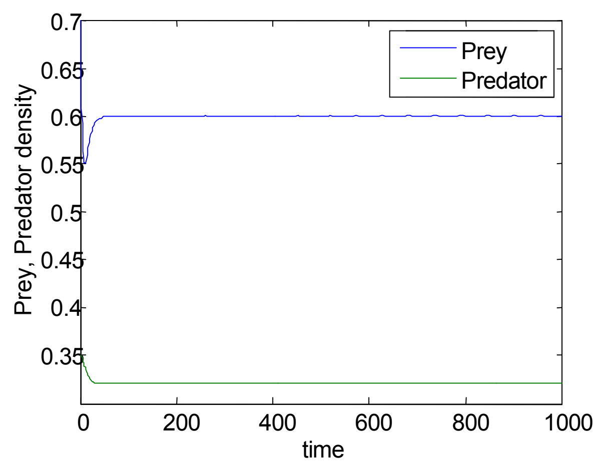

Fig. 6: Time series plot of prey and predator at γ=0.04

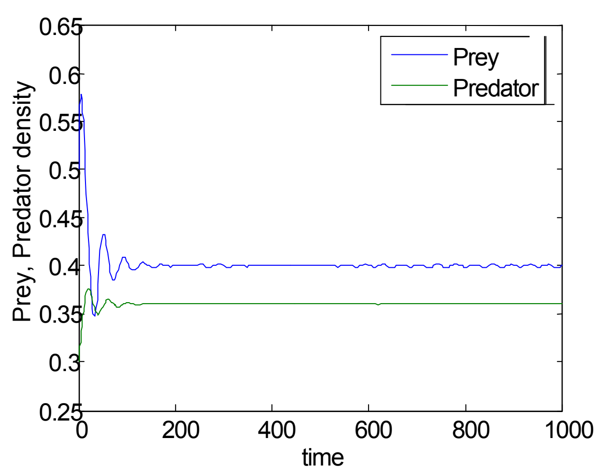

Fig. 7: Time series plot of prey and predator at γ=0.06

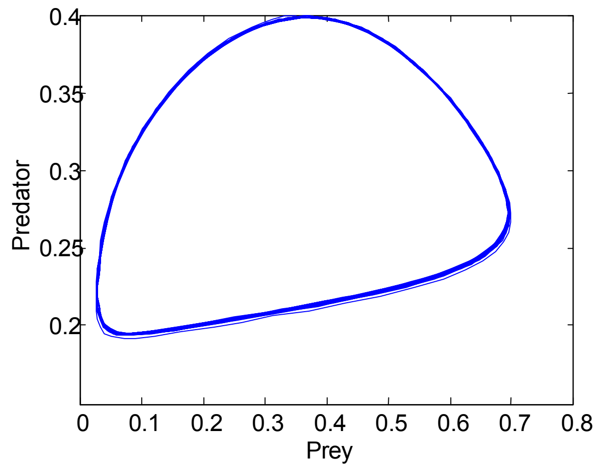

Fig. 8: Phase portrait of prey and predator at γ=0.07.

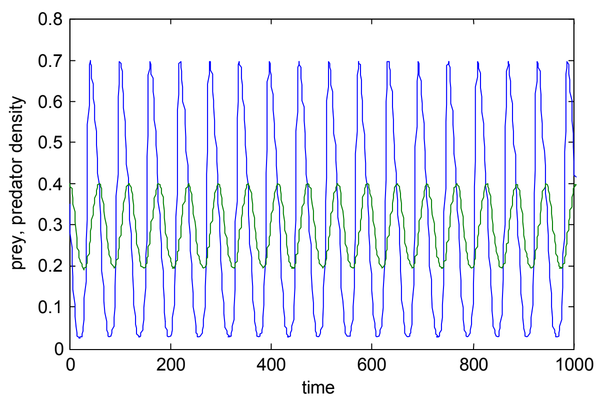

Fig. 9: Time series plot of prey and predator at γ=0.07.

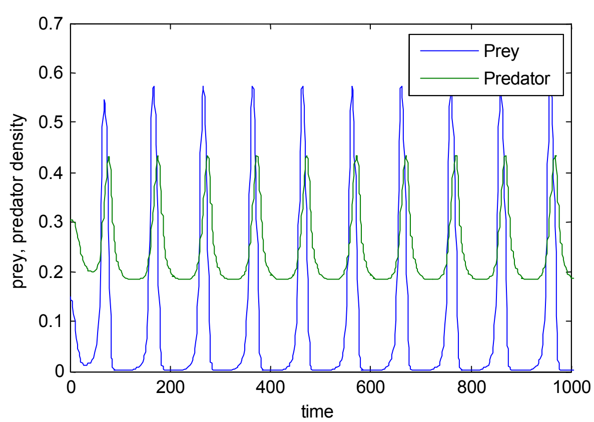

Fig. 10: Time series plot of prey and predator at γ=0.09.

CONCLUSION

This paper presents a prey-predator model with Holling type II functional response and modified Leslie Gower incorporating a constant proportion of prey refuge. Incorporating a refuge into system (4) provides a more realistic model. Refugi, therefore, can be considered as, areas in which the predator is not successfully controlling the prey and important for the biological control of a predator. The main focus of this paper was to introduce new mathematical models of biological systems and techniques for their analysis. Local asymptotic stability of the positive equilibrium implies its global asymptotic stability. Moreover, we established some new results such as the existence of stable or unstable equilibrium points under suitable values of parameters in the models. Two species can coexist in the case of stable condition; otherwise they might be extinct in the case of unstable condition.

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