European Journal of Applied Sciences 4 (2): 78-86, 2012
ISSN 2079-2077

Variational Iteration Method for System of Higher-Order BVPs

Muhammad Shakeel, Qazi Mahmood ul Hassan and Syed Tauseef Mohyud-Din

Department of Mathematics, HITEC University, Taxila Cantt Pakistan

Abstract: In this paper, we apply Variational Iteration Method (VIM) to solve eighth-order Boundary Value Problems (BVPs) which arise frequently in astronomy and other physical problems. Numerical results completely reveal the efficiency and reliability of the proposed VIM.

Key words: Variational iteration method . astronomy . higher-order BVPs

INTRODUCTION

In this paper, we consider the general eighth-order boundary value problem of the type

$$u^{(8)}(x)+\phi(x)u(x)=\psi(x),\ a\le x\le b \qquad (1)$$

with boundary conditions

$$u(a)=A_0,\ u^{(2)}(a)=A_2,\ u^{(4)}(a)=A_4,\ u^{(6)}(a)=A_6$$

$$u(b)=B_0,\ u^{(2)}(b)=B_2,\ u^{(4)}(b)=B_4,\ u^{(6)}(b)=B_6 \qquad (2)$$

where $u(x)$, $\phi(x)$ and $\psi(x)$ are continuous functions defined on $[a, b]$ and the constants A_i and B_i are finite real numbers. These problems arise frequently in hydrodynamic, hydromagnetic stability and astronmy [6, 10, 11] and references therein. The basic motivation of this paper is the extension of Variational Iteration Method (VIM) to solve re-formulated eight-order boundary value probems. The eighth-order BVPs are converted into an equivalent system of integral equations by using a suitable transformation. The VIM [1-10] is applied to the re-formulated system of integral equations. Numerical results are very encouraging. Several examples are given to illustrate the reliability and performance of the proposed algorithm (MVIM).

VARIATIONAL ITERATION METHOD

To illustrate the basic concept of the technique, we consider the following general differential equation:

$$Lu+Nu=g(x) \qquad (3)$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the forcing term. According to variational iteration method [1-10], we can construct a correction functional as follows:

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(s)\left(Lu_n(s)+N\tilde{u}_n(s)-g(s)\right)ds \qquad (4)$$

where λ is a Lagrange multiplier [1-10], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta\tilde{u}_n=0$; (4) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given [16-19]. For the sake of simplicity and to convey the idea of the technique, we consider the following system of differential equations:

$$x_i'(t)=f_i(t,x_i),\quad i=1,2,3,\ldots,n \qquad (5)$$

subject to the boundary conditions

$$x_i(0)=c_i,\quad i=1,2,3,\ldots,n \qquad (6)$$

To solve the system by means of the variational iteration method, we rewrite the system (5) in the following form:

$$x_i'(t)=f_i(x_i)+g_i(t),\quad i=1,2,3,\ldots,n \qquad (7)$$

subject to the boundary conditions $x_i(0)=c_i$, $i=1,2,3,\ldots,n$ and g_i is defined in (3).

The correction functional for the nonlinear system (5) can be approximated as

Corresponding Author: Dr. Syed Tauseef Mohyud-Din, Department of Mathematics, HITEC University Taxila Cantt Pakistan.

$$x_1^{(k+1)}(t) = x_1^{(k)}(t) + \int_0^t \lambda_1 \left(x_1^{'(k)}(T), f_1\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)\right) - g_1(T)\right) dT$$
$$x_2^{(k+1)}(t) = x_2^{(?a)}(t) + \int_0^t \lambda_2 \left(x_2^{'(k)}(T), f_2\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)\right) - g_2(T)\right) dT$$
.
.
.
$$x_n^{(k+1)}(t) = x_n^{(k)}(t) + \int_0^t \lambda_n \left(x_n^{'(k)}(T), f_n\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)\right) - g_n(T)\right) dT$$

where $\lambda_i = \pm 1$, i = 1,2,3,…,n are Lagrange multipliers, $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n$ denote the restricted variations. For λ_i = -1, i = 1,2,3,…,n, we have the following iterative scheme:

$$x_1^{(k+1)}(t) = x_1^{(k)}(t) - \int_0^t \lambda_1 \left(x_1^{'(k)}(T), f_1\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)\right) - g_1(T)\right) dT$$
$$x_2^{(k+1)}(t) = x_2^{(k)}(t) - \int_0^t \lambda_2 \left(x_2^{'(k)}(T), f_2\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)\right) - g_2(T)\right) dT$$
.
.
.
$$x_n^{(k+1)}(t) = x_n^{(k)}(t) - \int_0^t \lambda_n \left(x_n^{'(k)}(T), f_n\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)\right) - g_n(T)\right) dT$$

If we start with the initial approximations $x_i(0) = c_i$, i = 1,2,3,…,n, then the approximations can be completely determined; finally we approximate the solution

$$x_i(t) = \lim_{k\to\infty} x_i^{(k)}$$

by the nth term $x_i^{(n)}(t)$ for i = 1,2,3,…,n (8)

NUMERICAL APPLICATIONS

In this section, we first rewrite the eighth-order boundary value problem in an equivalent system of integral equations by using a suitable transformation. The VIM is applied to the re-formulated system of integral equations. Numerical results are very encouraging.

Example 3.1: Consider the Eq. (1) with

$$[a, b] = [0, 1], \phi(x) = x$$

and

$$\psi(x) = -\left(48 + 15x + x^3\right)e^x$$

and the boundary conditions

$$A_0 = 0, A_2 = 0, A_4 = -8, A_6 = -24$$

$$B_0 = 0, B_2 = -4e, B_4 = -16e, B_6 = -36e \quad (9)$$

The exact solution is given by

$$x(1-x)e^x \quad (10)$$

Using the transformation

$$\frac{du}{dx} = a(x), \frac{da}{dx} = b(x), \frac{db}{dx} = e(x), \frac{de}{dx} = f(x)$$

$$\frac{df}{dx} = g(x), \frac{dg}{dx} = h(x), \frac{dh}{dx} = z(x) \quad (11)$$

we obtain the following system of differential equations:

$$\frac{du}{dx} = a(x), \frac{da}{dx} = b(x), \frac{db}{dx} = e(x), \frac{de}{dx} = f(x)$$
$$\frac{df}{dx} = g(x), \frac{dg}{dx} = h(x), \frac{dh}{dx} = z(x),$$
$$\frac{dz}{dx} = \left(-xu(x) - \left(48 + 15x + x^3\right)e^x\right) \quad (12)$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers λ_i = 1, i = 1,2,3,…,8:

$$u^{(k+1)} = 0 + \int_0^x a^{(k)}(s)ds,\ a^{(k+1)} = A + \int_0^x b^{(k)}(s)ds$$

$$b^{(k+1)} = 0 + \int_0^x e^{(k)}(s)ds, e^{(k+1)} = B + \int_0^x f^{(k)}(s)ds$$

$$f^{(k+1)} = -8 + \int_0^x g^{(k)}(s)ds, g^{(k+1)} = C + \int_0^x h^{(k)}(s)ds$$

$$h^{(k+1)} = -24 + \int_0^x z^{(k)}(s)ds$$

$$z^{(k+1)} = L + \int_0^x \left(-su(s) - \left(48 + 15s + s^3\right)e^s\right)ds \quad (13)$$

Consequently, we obtain the following approximants:

$$u^{(0)}(x) = 0, a^{(0)}(x) = A, b^{(0)}(x) = 0, e^{(0)}(x) = B$$

$$f^{(0)}(x) = -8, g^{(0)}(x) = C, h^{(0)}(x) = -24, z^{(0)}(x) = L$$

$$u^{(1)}(x) = Ax, a^{(1)}(x) = A$$

$$b^{(1)}(x) = Bx, e^{(1)}(x) = B - 8x$$

$$f^{(1)}(x) = -8 + Cx, g^{(1)}(x) = C - 24x$$

$$h^{(1)}(x) = -24 + Lx$$

$$z^{(1)}(x) = L + 27 - 27e^x - 21e^x x + 3e^x x^2 - e^x x^3$$

$$u^{(2)}(x) = Ax, a^{(2)}(x) = A + \frac{1}{2}Bx^2$$

$$b^{(2)}(x) = Bx - 4x^2, e^{(2)}(x) = B - 8x + \frac{1}{2}Cx^2$$

$$f^{(2)}(x) = -8 + Cx - 12x^2$$

$$g^{(2)}(x) = C - 24x + \frac{1}{2}Lx^2$$

$$h^{(2)}(x) = -30 + Lx + 27x + 6e^x - 33e^x x + 6e^x x^2 - e^x x^3$$

$$z^{(2)}(x) = L + 27 - 27e^x - 21e^x x + 3e^x x^2 - e^x x^3 - \frac{1}{3}x^3 A$$

$$u^{(3)}(x) = Ax + \frac{1}{6}Bx^3$$

$$a^{(3)}(x) = A + \frac{1}{2}Bx^2 - \frac{4}{3}x^3$$

$$b^{(3)}(x) = Bx - 4x^2 + \frac{1}{6}Cx^3$$

$$e^{(3)}(x) = B - 8x + \frac{1}{2}Cx^2 - 4x^3$$

$$f^{(3)}(x) = -8 + Cx - 12x^2 + \frac{1}{6}Lx^3$$

$$g^{(3)}(x) = C - 57 - 30x + \frac{1}{2}Lx^2 + \frac{27}{2}x^2 + 57e^x - 51e^x x + 9e^x x^2 - e^x x^3$$

$$h^{(3)}(x) = -30 + Lx + 27x + 6e^x - 33e^x x + 6e^x x^2 - e^x x^3 - \frac{1}{12}Ax^4$$

$$z^{(3)}(x) = L + 27 - 27e^x - 21e^x x - e^x x^3 + 3e^x x^2 - \frac{1}{3}x^3 A \quad (14)$$

.
.
.

The series solution is given by

$$u(x) = Ax + \frac{1}{6}Bx^3 - \frac{1}{3}x^4 + \frac{1}{120}Cx^5 - \frac{1}{30}x^6 + \frac{1}{5040}Lx^7 - \frac{1}{840}x^8 - \frac{1}{5760}x^9 - \left(\frac{13}{604800} + \frac{1}{1814400}A\right)x^{10} - \frac{1}{403200}x^{11}$$
$$-\left(\frac{1}{3628800} + \frac{1}{119750400}B\right)x^{12} - \frac{1}{43545600}x^{13} - \left(\frac{43}{14529715200} + \frac{1}{14529715200}C\right)x^{14} - \frac{1}{6706022400}x^{15} + O\left(x^{16}\right) \quad (15)$$

Imposing the boundary conditions at $x = 1$, we have

$$A = 1.000000131,\ B = -3.000001255,\ C = -14.99998905,\ L = -35.00006877 \quad (16)$$

Consequently, the series solution is given as

$$u(x) = 1.000000131x - 0.5000002092x^3 - \frac{1}{3}x^4 - 0.1249999088x^5 - \frac{1}{30}x^6 - 0.006944458089x^7 -$$
$$\frac{1}{840}x^8 - \frac{1}{5760}x^9 - 2.204585545\times10^{-5}x^{10} - \frac{1}{403200}x^{11} - 2.505210733\times10^{-7}x^{12}$$
$$-\frac{1}{43545600}x^{13} - 1.927086014\times10^{-9}x^{14} - \frac{1}{6706022400}x^{15} + O\left(x^{16}\right) \quad (17)$$

Table 3.1 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $u(x)$.

Table 3.1: Error estimates

x	Exact solution	Series solution	*Error
0.0	0.0000000000	0.0000000000	0.00000000000+00
0.1	0.0994653955	0.0994653826	1.2910000000E-08
0.2	0.1954244659	0.1954244413	2.4600000000E-08
0.3	0.2834703834	0.2834703497	3.3700000000E-08
0.4	0.3580379674	0.3580379275	3.9900000000E-08
0.5	0.4121803599	0.4121803178	4.2100000000E-08
0.6	0.4373085523	0.4373085120	4.0300000000E-08
0.7	0.4228881028	0.4228880685	3.4300000000E-08
0.8	0.3560865733	0.3560865485	2.4800000000E-08
0.9	0.2213642927	0.2213642800	1.2700000000E-08
1.0	-0.0000000006	0.0000000000	5.9599169680E-10

*Error = Exact solution-Series solution

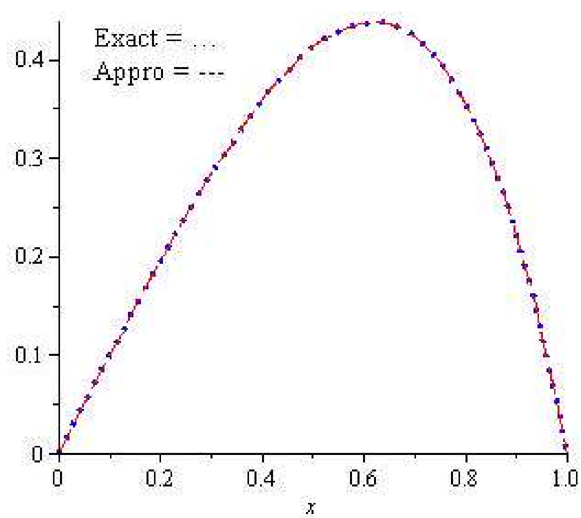

Fig.1:

Example 3.2: Consider the Eq. (1) with

$$[a,b]=[0,1],\ \phi(x)=-x$$

and

$$\psi(x)=-\left(55+17x+x^2-x^3\right)e^x$$

and the boundary conditions

$$\begin{aligned}&A_0=0,\ A_2=2/e,\ A_4=-4/e,\ A_6=-18/e\\&B_0=0,\ B_2=-6e,\ B_4=-20e,\ B_6=-42e\end{aligned}\qquad(18)$$

The exact solution is given by

$$\left(1-x^2\right)e^x \qquad (19)$$

Using the transformation

$$\begin{aligned}&\frac{du}{dx}=a(x),\ \frac{da}{dx}=b(x),\ \frac{db}{dx}=e(x),\ \frac{de}{dx}=f(x)\\&\frac{df}{dx}=g(x),\ \frac{dg}{dx}=h(x),\ \frac{dh}{dx}=z(x)\end{aligned}\qquad(20)$$

we obtain the following system of differential equations:

$$\begin{aligned}&\frac{du}{dx}=a(x),\ \frac{da}{dx}=b(x),\ \frac{db}{dx}=e(x),\ \frac{de}{dx}=f(x)\\&\frac{df}{dx}=g(x),\ \frac{dg}{dx}=h(x),\ \frac{dh}{dx}=z(x),\\&\frac{dz}{dx}=\left(xu(x)-\left(55+17x+x^2-x^3\right)e^x\right)\end{aligned}\qquad(21)$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1$, $i = 1,2,3,\ldots,8$:

$$\begin{aligned}&u^{(k+1)}=A+\int_0^x a^{(k)}(s)ds,\ a^{(k+1)}=B+\int_0^x b^{(k)}(s)ds\\&b^{(k+1)}=C+\int_0^x e^{(k)}(s)ds,\ e^{(k+1)}=E+\int_0^x f^{(k)}(s)ds\\&f^{(k+1)}=F+\int_0^x g^{(k)}(s)ds,\ g^{(k+1)}=G+\int_0^x h^{(k)}(s)ds\\&h^{(k+1)}=H+\int_0^x z^{(k)}(s)ds\\&z^{(k+1)}=L+\int_0^x\left(su(s)-\left(55+17s+s^2-s^3\right)e^s\right)ds\end{aligned}\qquad(22)$$

Consequently, we obtain the following approximants:

$$\begin{aligned}&u^{(0)}(x)=A,\ a^{(0)}(x)=B,\ b^{(0)}(x)=C,\ e^{(0)}(x)=E\\&f^{(0)}(x)=F,\ g^{(0)}(x)=G,\ h^{(0)}(x)=h,\ z^{(0)}(x)=L\\&u^{(1)}(x)=A+Bx,\ a^{(1)}(x)=B+Cx\\&b^{(1)}(x)=C+Ex,\ e^{(1)}(x)=E+Fx\\&f^{(1)}(x)=F+Gx,\ g^{(1)}(x)=G+Hx\\&h^{(1)}(x)=H+Lx,\ z^{(1)}(x)\\&\qquad=L+46+\frac{1}{2}x^2A-46e^x-9e^xX-4e^xx^2+e^xx^3\\&u^{(2)}(x)=A+Bx+\frac{1}{2}Cx^2,\ a^{(2)}(x)=B+Cx+\frac{1}{2}Ex^2\\&b^{(2)}(x)=C+Ex+\frac{1}{2}Fx^2,\ e^{(2)}(x)=E+Fx+\frac{1}{2}Gx^2\\&f^{(2)}(x)=F+Gx+\frac{1}{2}Hx^2,\ g^{(2)}(x)=G+Hx+\frac{1}{2}Lx^2\\&h^{(2)}(x)=H+Lx+51+46x+\frac{1}{6}x^3A\\&\qquad-51e^x+5e^xx-7e^xx^2+e^xx^3\end{aligned}$$

$$z^{(2)}(x) = L + 46 + \frac{1}{2}x^2A - 46e^x - 9e^x x - 4e^x x^2 + e^x x^3 + \frac{1}{3}x^3B$$

$$u^{(3)}(x) = A + Bx + \frac{1}{2}Cx^2 + \frac{1}{6}Ex^3$$

$$a^{(3)}(x) = B + Cx + \frac{1}{2}Ex^2 + \frac{1}{6}Fx^3$$

$$b^{(3)}(x) = C + Ex + \frac{1}{2}Fx^2 + \frac{1}{6}Gx^3$$

$$e^{(3)}(x) = E + Fx + \frac{1}{2}Gx^2 + \frac{1}{6}Hx^3$$

$$f^{(3)}(x) = F + Gx + \frac{1}{2}Hx^2 + \frac{1}{6}Lx^3$$

$$g^{(3)}(x) = G + Hx + \frac{1}{2}Lx^2 + 76 + 51x + 23x^2 + \frac{1}{24}x^4A - 76e^x + 25e^x x - 10e^x x^2 + e^x x^3$$

$$h^{(3)}(x) = H + Lx + 51 + 46x + \frac{1}{6}x^3A - 51e^x + 5e^x x - 7e^x x^2 + e^x x^3 + \frac{1}{12}Bx^4$$

$$z^{(3)}(x) = L + 46 + \frac{1}{2}x^2A - 46e^x - 9e^x x - 4e^x x^2 + e^x x^3 + \frac{1}{3}x^3B + \frac{1}{8}x^4C \qquad (23)$$

$$\vdots$$

The series solution is given by

$$u(x) = A + Bx + \frac{1}{2}Cx^2 + \frac{1}{6}Ex^3 + \frac{1}{24}Fx^4 + \frac{1}{120}Gx^5 + \frac{1}{720}Hx^6 + \frac{1}{5040}Lx^7 - \frac{11}{8064}x^8 - \left(\frac{1}{362880}A - \frac{1}{5040}\right)x^9 + \left(-\frac{13}{518400} + \frac{1}{1814400}B\right)x^{10} + \left(-\frac{53}{19958400} + \frac{1}{13305600}C\right)x^{11} + \left(\frac{1}{119750400}E - \frac{37}{159667200}\right)x^{12} + \left(-\frac{1}{62270208} + \frac{1}{1245404160}F\right)x^{13} + \left(-\frac{67}{87178291200} + \frac{1}{14529715200}G\right)x^{14} + \left(-\frac{1}{217945728000} + \frac{1}{186810624000}H\right)x^{15} + O\left(x^{16}\right) \qquad (24)$$

Imposing the boundary conditions at x =-1 and x =1, we have

$$A = 1.000005531,\ B = 1.000000014,\ C = -1.000013606,\ E = -5.000000133$$
$$F = -10.99996725,\ G = -18.99999884,\ H = -29.00006650,\ L = -41.00000746 \qquad (25)$$

Consequently, the series solution is given as

$$u(x) = 1.000005531 + 1.000000014x - 0.5000068030x^2 - 0.8333333555 - 0.4583319688x^4 - 0.1583333237x^5 - 0.04027787014x^6 - 0.008134922115x^7 - 0.0013640873x^8 - 0.0001956569512x^9 - 0.00002452601410x^{10} - 0.000002730680837x^{11} - 2.734855177\times10^{-7}x^{12} - 2.489149165\times10^{-8}x^{13} - 2.076204873\times10^{-9}x^{14} - 1.598260779\times10^{-10}x^{15} + O\left(x^{16}\right) \qquad (26)$$

Table 3.2 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $u(x)$.

Example 3.3: Consider the Eq. (1) with

$$[a, b] = [-1, 1],\ \phi(x) = -1 \ \text{and}\ \psi(x) = -8\left[2x\cos(x) + 7\sin(x)\right]$$

and the boundary conditions

Table 3.2: Error estimates

x	Exact solution	Series solution	*Error
1.00	0.0000000002	0.0000000000	1.8053215490E-10
0.80	0.1617601361	0.1617584271	1.7090000000E-06
0.60	0.3512426973	0.3512394471	3.2502000000E-06
0.40	0.5630733113	0.5630688386	4.4727000000E-06
0.20	0.7859867813	0.7859815230	5.2583000000E-06
0.00	1.0000055310	1.0000000000	5.5310000000E-06
0.20	1.1725519120	1.1725466480	5.2640000000E-06
0.40	1.2531372290	1.2531327460	4.4830000000E-06
0.60	1.1661592910	1.1661560320	3.2590000000E-06
0.80	0.8011964484	0.8011947341	1.7143000000E-06
1.00	0.0000000002	0.0000000000	1.8382269910E-10

*Error = Exact solution--Series solution

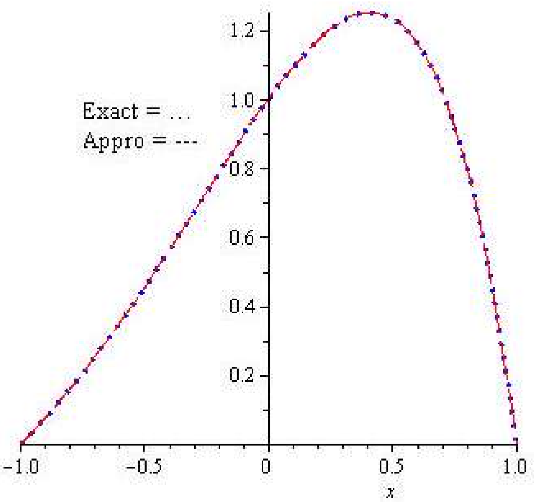

Fig. 2:

$$A_0 = 0, A_2 = -4\cos(1) - 2\sin(1)$$
$$A_4 = 8\cos(1) + 12\sin(1), A_6 = -12\cos(1) - 30\sin(1)$$
$$B_0 = 0, B_2 = 4\cos(1) + 2\sin(1)$$
$$B_4 = -8\cos(1) - 12\sin(1), B_6 = 12\cos(1) + 30\sin(1) \quad (27)$$

The exact solution is given by

$$\left(x^2 - 1\right)\sin(x) \quad (28)$$

Using the transformation

$$\frac{du}{dx} = a(x), \frac{da}{dx} = b(x), \frac{db}{dx} = e(x), \frac{de}{dx} = f(x)$$
$$\frac{df}{dx} = g(x), \frac{dg}{dx} = h(x), \frac{dh}{dx} = z(x) \quad (29)$$

we obtain the following system of differential equations:

$$\frac{du}{dx} = a(x), \frac{da}{dx} = b(x), \frac{db}{dx} = e(x), \frac{de}{dx} = f(x)$$
$$\frac{df}{dx} = g(x), \frac{dg}{dx} = h(x), \frac{dh}{dx} = z(x),$$
$$\frac{dz}{dx} = \left(u(x) - 8\left[2x\cos(x) + 7\sin(x)\right]\right) \quad (30)$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1, i = 1,2,3,\ldots,8$:

$$u^{(k+1)} = A + \int_0^x a^{(k)}(s)ds,\ a^{(k+1)} = B + \int_0^x b^{(k)}(s)ds$$
$$b^{(k+1)} = C + \int_0^x e^{(k)}(s)ds, e^{(k+1)} = E + \int_0^x f^{(k)}(s)ds$$
$$f^{(k+1)} = F + \int_0^x g^{(k)}(s)ds, g^{(k+1)} = G + \int_0^x h^{(k)}(s)ds$$
$$h^{(k+1)} = H + \int_0^x z^{(k)}(s)ds$$
$$z^{(k+1)} = L + \int_0^x \left(u(s) - 8\left[2s\cos(s) + 7\sin(s)\right]\right)ds \quad (31)$$

Consequently, we obtain the following approximants:

$$u^{(0)}(x) = A,\ a^{(0)}(x) = B$$
$$b^{(0)}(x) = C, e^{(0)}(x) = E$$
$$f^{(0)}(x) = F, g^{(0)}(x) = G$$
$$h^{(0)}(x) = H,\ z^{(0)}(x) = L$$
$$u^{(1)}(x) = A + Bx, a^{(1)}(x) = B + Cx$$
$$b^{(1)}(x) = C + Ex,\ e^{(1)}(x) = E + Fx$$
$$f^{(1)}(x) = F + Gx, g^{(1)}(x) = G + Hx$$
$$h^{(1)}(x) = H + Lx, z^{(1)}(x) = L - 40 + Ax + 40\cos x - 16x\sin x$$
$$u^{(2)}(x) = A + Bx + \frac{1}{2}Cx^2,\ a^{(2)}(x) = B + Cx + \frac{1}{2}Ex^2$$
$$b^{(2)}(x) = C + Ex + \frac{1}{2}Fx^2, e^{(2)}(x) = E + Fx + \frac{1}{2}Gx^2$$
$$f^{(2)}(x) = F + Gx + \frac{1}{2}Hx^2, g^{(2)}(x) = G + Hx + \frac{1}{2}Lx^2$$
$$h^{(2)}(x) = H + Lx - 40x + \frac{1}{2}Ax^2 + 24\sin(x) + 16x\cos(x)$$
$$z^{(2)}(x) = L - 40 + Ax + 40\cos(x) - 16\sin x(x) + \frac{1}{2}Bx^2$$
$$u^{(3)}(x) = A + Bx + \frac{1}{2}Cx^2 + \frac{1}{6}Ex^3$$
$$a^{(3)}(x) = B + Cx + \frac{1}{2}Ex^2 + \frac{1}{6}Fx^3$$
$$b^{(3)}(x) = C + Ex + \frac{1}{2}Fx^2 + \frac{1}{6}Gx^3$$

$$e^{(3)}(x) = E + Fx + \frac{1}{2}Gx^2 + \frac{1}{6}Hx^3$$
$$f^{(3)}(x) = F + Gx + \frac{1}{2}Hx^2 + \frac{1}{6}Lx^3$$
$$g^{(3)}(x) = G + Hx + \frac{1}{2}Lx^2 + 8 - 20x^2 + \frac{1}{6}Ax^3 - 8\cos(x) + 16x\sin(x)$$
$$h^{(3)}(x) = H + Lx - 40x + \frac{1}{2}Ax^2 + 24\sin(x) + 16x\cos(x) + \frac{1}{6}Bx^3$$
$$z^{(3)}(x) = L - 40 + Ax + 24\sin(x) + 16x\cos(x) + \frac{1}{6}Bx^3 \quad (32)$$

The series solution is given by

$$u(x) = A + Bx + \frac{1}{2}Cx^2 + \frac{1}{6}Ex^3 + \frac{1}{24}Fx^4 + \frac{1}{120}Gx^5 + \frac{1}{720}Hx^6 + \frac{1}{5040}Lx^7 + \frac{1}{40320}Ax^8 + \left(-\frac{1}{5040} + \frac{1}{362880}B\right)x^9$$
$$+ \frac{1}{3628800}Cx^{10} + \left(\frac{13}{4989600} + \frac{1}{39916800}E\right)x^{11} + \frac{1}{479001600}Fx^{12} + \left(-\frac{17}{778377600} + \frac{1}{6227020800}G\right)x^{13}$$
$$+ \frac{1}{87178291200}Hx^{14} + \left(\frac{1}{7783776000} + \frac{1}{1307674368000}L\right)x^{15} + O\left(x^{16}\right)$$

Imposing the boundary conditions at $x = -1$ and $x = 1$, we have

$$A = 0,\ B = -0.9999999860,\ C=0,\ E=6.999999872,\ F=0,\ G= -20.99999890,\ H=0,\ L= -41.00000746 \quad (33)$$

Consequently, the series solution is given as

$$u(x) = -0.9999999860x + 1.166666645x^3 - 0.1749999908x^5 + 0.008531744685x^7 - 0.0002011684303x^9$$
$$+ 0.000002780784028x^{11} - 2.52126986510^{-8}x^{13} + 1.61355149610^{-10}x^{15} + O\left(x^{16}\right) \quad (34)$$

Table 3.3: Error estimates

x	Exact solution	Series solution	*Error
-1.00	-0.0000000002	0.0000000000	1.8738449960E-10
-0.80	0.2582481900	0.2582481927	2.7000000000E-09
-0.60	0.3613711786	0.3613711830	4.4000000000E-09
-0.40	0.3271114032	0.3271114075	4.3000000000E-09
-0.20	0.1907225549	0.1907225576	2.7000000000E-09
0.00	0.0000000000	0.0000000000	0.00000000000000
0.20	-0.1907225549	-0.1907225576	2.7000000000E-09
0.40	-0.3271114032	-0.3271114075	4.3000000000E-09
0.60	-0.3613711786	-0.3613711830	4.4000000000E-09
0.80	-0.2582481900	-0.2582481927	2.7000000000E-09
1.00	0.0000000002	0.0000000000	1.8738449960E-10

*Error = Exact solution-Series solution

Table 3.3 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $u(x)$.

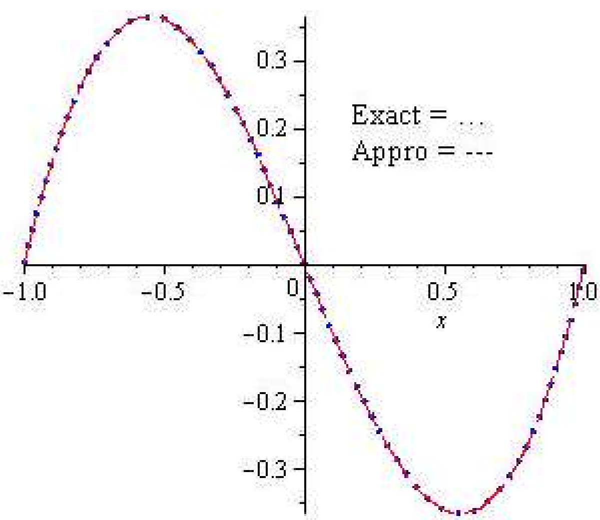

Fig. 3:

Example 3.4: Consider the Eq. (1) with

$$[a,b] = [-1,1],\ \phi(x) = -1$$

and

$$\psi(x) = 8\left[2x\sin(x) - 7\cos(x)\right]$$

and the boundary conditions

$$\begin{aligned}
&A_0 = 0, A_2 = -4\sin(1) + 2\cos(1) \\
&A_4 = 8\sin(1) - 12\cos(1),\ A_6 = -12\sin(1) + 30\cos(1) \\
&B_0 = 0, B_2 = -4\sin(1) + 2\cos(1) \\
&B_4 = 8\sin(1) - 12\cos(1),\ B_6 = -12\sin(1) + 30\cos(1) \qquad (35)
\end{aligned}$$

The exact solution is given by

$$\left(x^2 - 1\right)\cos(x) \qquad (36)$$

Using the transformation

$$\begin{aligned}
&\frac{du}{dx} = a(x), \frac{da}{dx} = b(x), \frac{db}{dx} = e(x), \frac{de}{dx} = f(x) \\
&\frac{df}{dx} = g(x), \frac{dg}{dx} = h(x), \frac{dh}{dx} = z(x) \qquad (37)
\end{aligned}$$

we obtain the following system of differential equations:

$$\begin{aligned}
&\frac{du}{dx} = a(x), \frac{da}{dx} = b(x), \frac{db}{dx} = e(x), \frac{de}{dx} = f(x) \\
&\frac{df}{dx} = g(x), \frac{dg}{dx} = h(x), \frac{dh}{dx} = z(x), \\
&\frac{dz}{dx} = \left(u(x) + 8\left[2x\sin(x) - 7\cos(x)\right]\right) \qquad (38)
\end{aligned}$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1$, $i = 1,2,3,\ldots,8$:

$$\begin{aligned}
&u^{(k+1)} = A + \int_0^x a^{(k)}(s)ds,\ a^{(k+1)} = B + \int_0^x b^{(k)}(s)ds \\
&b^{(k+1)} = C + \int_0^x e^{(k)}(s)ds,\ e^{(k+1)} = E + \int_0^x f^{(k)}(s)ds \\
&f^{(k+1)} = F + \int_0^x g^{(k)}(s)ds, g^{(k+1)} = G + \int_0^x h^{(k)}(s)ds \\
&h^{(k+1)} = H + \int_0^x z^{(k)}(s)ds \\
&z^{(k+1)} = L + \int_0^x \left(u(s) + 8\left[2x\sin(x) - 7\cos(x)\right]\right)ds \qquad (39)
\end{aligned}$$

Consequently, we obtain the following approximants:

$$\begin{aligned}
&u^{(0)}(x) = A, a^{(0)}(x) = B \\
&b^{(0)}(x) = C,\ e^{(0)}(x) = E \\
&f^{(0)}(x) = F,\ g^{(0)}(x) = G \\
&h^{(0)}(x) = H, z^{(0)}(x) = L \\
&u^{(1)}(x) = A + Bx, a^{(1)}(x) = B + Cx \\
&b^{(1)}(x) = C + Ex,\ e^{(1)}(x) = E + Fx \\
&f^{(1)}(x) = F + Gx,\ g^{(1)}(x) = G + Hx \\
&h^{(1)}(x) = H + Lx,\ z^{(1)}(x) = L + Ax + 40\sin x - 16x\cos x \\
&u^{(2)}(x) = A + Bx + \frac{1}{2}Cx^2,\ a^{(2)}(x) = B + Cx + \frac{1}{2}Ex^2 \\
&b^{(2)}(x) = C + Ex + \frac{1}{2}Fx^2,\ e^{(2)}(x) = E + Fx + \frac{1}{2}Gx^2 \\
&f^{(2)}(x) = F + Gx + \frac{1}{2}Hx^2,\ g^{(2)}(x) = G + Hx + \frac{1}{2}Lx^2 \\
&h^{(2)}(x) = H + Lx - 24 + \frac{1}{2}Ax^2 + 16x\cos x + 16x\sin x \\
&z^{(2)}(x) = L - 40\sin x - 16x\cos x + \frac{1}{2}Bx^2 \\
&u^{(3)}(x) = A + Bx + \frac{1}{2}Cx^2 + \frac{1}{6}Ex^3 \\
&a^{(3)}(x) = B + Cx + \frac{1}{2}Ex^2 + \frac{1}{6}Fx^3 \\
&b^{(3)}(x) = C + Ex + \frac{1}{2}Fx^2 + \frac{1}{6}Gx^3 \\
&e^{(3)}(x) = E + Fx + \frac{1}{2}Gx^2 + \frac{1}{6}Hx^3 \\
&f^{(3)}(x) = F + Gx + \frac{1}{2}Hx^2 + \frac{1}{6}Lx^3 \\
&g^{(3)}(x) = G + Hx + \frac{1}{2}Lx^2 - 24x + \frac{1}{6}Ax^3 + 8\sin x + 16x\cos x \\
&h^{(3)}(x) = H + Lx - 24 + \frac{1}{2}Ax^2 + 24\cos(x) - 16x\sin(x) + \frac{1}{6}Bx^3 \\
&z^{(3)}(x) = L + Ax - 40\sin(x) - 16x\cos(x) + \frac{1}{2}Bx^3 + \frac{1}{2}Cx^3 \qquad (40)
\end{aligned}$$

.
.
.

The series solution is given by

$$\begin{aligned}
u(x) = {} & A + Bx + \frac{1}{2}Cx^2 + \frac{1}{6}Ex^3 + \frac{1}{24}Fx^4 + \frac{1}{120}Gx^5 + \frac{1}{720}Hx^6 + \frac{1}{5040}Lx^7 + \left(-\frac{1}{720} + \frac{1}{40320}A\right)x^8 + \frac{1}{362880}Bx^9 \\
&+ \left(\frac{11}{453600} + \frac{1}{3626800}C\right)x^{10} + \frac{1}{39916800}Ex^{11} + \left(\frac{11}{479001600}F - \frac{1}{39916800}\right)x^{12} \\
&+ \frac{1}{6227020800}Gx^{13} + \left(\frac{19}{10897286400} + \frac{1}{87178291200}H\right)x^{14} + \frac{1}{1307674368000}Lx^{15} + O\left(x^{16}\right) \qquad (41)
\end{aligned}$$

Imposing the boundary conditions at $x = -1$ and $x = 1$, we have

$$A = -0.9999943757,\ B=0,\ C=2.999986164,\ E=0$$
$$F = -12.99996668,\ G=0,\ H=30.99993139,\ L=0 \quad (42)$$

Consequently, the series solution is given as

$$u(x) = 0.9999943757 + 1.499993082x^2 - 0.5416652783x^4 + 0.043055461x^6 - 0.001413690337x^8$$
$$+0.00002507715668x^{10} - 2.776607984\times10^{-7}x^{12} + 2.099145664\times10^{-9}x^{14} + O\left(x^{16}\right) \quad (43)$$

Table 3.4: Error estimates

x	Exact solution	Series solution	*Error
-1.00	0.0000000003	0.0000000000	2.6802726400E-10
-0.80	-0.2508126747	-0.2508144153	1.7406000000E-06
-0.60	-0.5282114843	-0.5282147935	3.3092000000E-06
-0.40	-0.7736866824	-0.7736912350	4.5526000000E-06
-0.20	-0.9408585649	-0.9408639147	5.3498000000E-06
0.00	-0.9999943757	-1.0000000000	5.6243000000E-06
0.20	-0.9408585649	-0.9408639147	5.3498000000E-06
0.40	-0.7736866824	-0.7736912350	4.5526000000E-06
0.60	-0.5282114843	-0.5282147935	3.3092000000E-06
0.80	-0.2508126747	-0.2508144153	1.7406000000E-06
1.00	0.0000000003	0.0000000000	2.6802726400E-10

*Error = Exact solution--Series solution

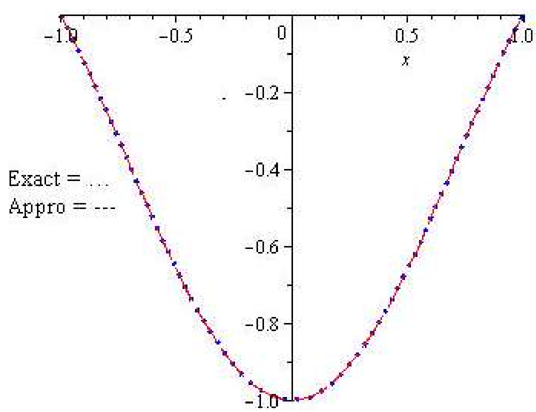

Fig. 4

Table 3.4 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $u(x)$.

CONCLUSION

In this paper, we applied Variational Iteration Method (VIM) for finding the analytical solutions of a class of eighth-order BVPs. Numerical results explicitly reveal the complete reliability of the proposed VIM.

REFERENCES

1. Abbasbandy, S., 2007. A new application of He's variational iteration method for quadratic Riccati differential equation by Adomian's polynomials. J. Comput. Appl. Math., 207: 59-63.
2. Abbasbandy, S., 2007. Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method. Internat. J. Numer. Meth. Engrg., 70: 876-881.
3. He, J.H., 2006. Some asymptotic methods for strongly nonlinear equation. Int. J. Mod. Phys., 20 (20): 1144-1199.
4. He, J.H., 2008. An elementary introduction of recently developed asymptotic methods and nanomechanics in textile engineering. Int. J. Mod. Phys. B 22 (21): 3487-4578.
5. Noor, M.A. and S.T. Mohyud-Din, 2008. Solution of singular and nonsingular initial and boundary value problems by modified variational iteration method. Mathematical Problems in Engineering, Hindawi, Article ID 917407, doi:10.1155/2008/91 7407, pp: 23.
6. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Some relatively new techniques for nonlinear problems. Mathematical Problems in Engineering, Hindawi, Article ID 234849, doi:10.1155/2009/23 4849, pp: 25.
7. Mohyud-Din, S.T., 2010. Variational iteration techniques for boundary value problems. VDM Verlag, ISBN 978-3-639-27664-0.
8. Mohyud-Din, S.T. and A. Yildirim, 2010. Latest developments in nonlinear sciences. Applications and Applied Mathematics, Special Issue 1: 46-72.
9. Yildirim, A., M.M. Hosseini, M. Usman and S.T. Mohyud-Din, 2010. On nonlinear sciences. Studies in Nonlinear Sciences, 1 (3): 97-117.
10. Siddiqi, S.S. and E.H. Twizell, 1996. Spline solution of linear eighth-order boundary value problems. Computer Methods in Applied Mechanics and Engineering, 131 (3-4): 309-325.
11. Wazwaz, A.-M., 2000. The numerical solution of special eight-order boundary value problems by the modified decomposition method. Neural, Parallel and Scientific Computations, 8 (2): 133-146.