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Homotopy Perturbation Method for Two-Dimensional Sine-Gordon Equation

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Abstract: In this paper, we employ homotopy perturbation method (HPM) to obtain olutions of two dimensional sine-Gordon equation which appears quite frequently in nonlinear and applied sciences. Numerical results explicilty reveal the complete reliability and efficiency of the proposed algorithm.

Key words: Two dimensional sine-Gordon equations · The homotopy perturbation method

INTRODUCTION

Nonlinear equations [1-25] appear frequently in the mathematical modeling of physical phenomenon related to applied and engineering sciences. Several techniques [1-10] including Backlund transformation, Darboux transformation, Adomian's decomposition, exp-function and variational iteration [5] have been used to find appropriate solutions of such equations, see [1-5] and the refernces therein. The basic motivation of this paper is the extension of applications of a very efficient technique namely Homotopy perturbation method (HPM) which is being applied on two dimensional Sine-Gordon (tdSG) equation with various initial values. Numerical results explicilty reveal the complete reliability and efficiency of the proposed algorithm.

Solution Procedure: Consider the following tdSG hyperbolic equation [10] which has the form

$$u_{tt} - u_{xx} - u_{yy} + m\sin(u) = 0,\ \ L_0 < x \le L_1,\ \ t > t_0, \qquad (1)$$

Where m is a constant. Initial conditions of Eq. (1) are assumed to have the form

$$u(x,y,t_0) = f(x,y),\quad u_1(x,y,t_0) = g(x,y),\quad L_0 \le x \le L_1 \qquad (2)$$

We construct the homotopy which satisfies the relation

$$u_{tt} + p\left(-u_{xx} - u_{yy} + m\sin(u)\right) = 0,\quad p \in [0,1] \qquad (3)$$

with initial conditions

$$u(x,y,0) = \arccos\left[\frac{\coth^2(K(x+dy))\left(1+\tanh^4(K(x+dy))\right)}{2}\right],$$

$$u_t(x,y,0) = \frac{cK\cosh(2K(x+dy))csh^3(K(x+dy))\sec h^3(K(x+dy))}{\sqrt{1-\dfrac{\coth^4(K(x+dy))\left(1+\tanh^2(K(x+dy))^4\right)}{4}}}, \qquad (4)$$

where c, d and K are arbitrary constants.
Assume the solution of Eq. (3) to be in the form:

$$y = y_0 + py_1 + p^2 y_2 + p^3 y_3 + ... \qquad (5)$$

Substituting Eq. (5) into Eq. (3) and equating the coefficients of like powers of p, we get following set of differential equations

$$p^0 : (u_0)_{tt} = 0, \qquad (6)$$

$$p^1 : (u_1)_{tt} - (u_0)_{xx} - (u_0)_{yy} + m^2\sin(u_0) = 0, \qquad (7)$$

$$p^2 : (u_2)_{tt} - (u_1)_{xx} - (u_1)_{yy} + m^2\cos(u_0) = 0, \qquad (8)$$

.....

The solution reads

$$u_0 = 0, \qquad (9)$$

$$u_1 = \arccos\left[\frac{\coth(K(x+dy))^2\left(1+\tanh(K(x+dy))^4\right)}{2}\right] \qquad (10)$$

$$+\frac{cKt\cosh(2K(x+dy))csh(K(x+dy))^3\sec h(K(x+dy))^3}{\sqrt{1-\dfrac{\coth(K(x+dy))^4\left(1+\tanh(K(x+dy))^4\right)^2}{4}}},$$

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$$u_2 = \frac{cKt^3 \cosh\left(2K\left(x+dy\right)\right)}{48\sqrt{\frac{-\cosh\left(K\left(x+dy\right)\right)^2 \csc h\left(K\left(x+dy\right)\right)^4 \sec h\left(K\left(x+dy\right)\right)^4}{4}}}[12K^2 + 12d^2K^2 + m^2$$

$$+\left(4K^2 + 4d^2K^2 - m^2\right)\cosh\left(4K\left(x+dy\right)\right)]\csc h\left(K\left(x+dy\right)\right)^5 \sec h\left(K\left(x+dy\right)\right)^5$$

$$+\frac{t^2 \csc h\left(K\left(x+dy\right)\right)^4 \sec h\left(K\left(x+dy\right)\right)^4}{256\sqrt{\frac{-\cosh\left(2K\left(x+dy\right)\right)^2 \csc h\left(K\left(x+dy\right)\right)^4 \sec h\left(K\left(x+dy\right)\right)^4}{4}}}\Big[-64K^2 - 64d^2K^2$$

$$-64K^2\cosh\left(4K\left(x+dy\right)\right) - 64d^2K^2\cosh\left(4K\left(x+dy\right)\right)$$

$$-3m^2 \arccos\left[\frac{\coth\left(K\left(x+dy\right)\right)^2\left(1+\tanh\left(K\left(x+dy\right)\right)^4\right)}{2}\right]$$

$$x\sqrt{\frac{-\cosh\left(2K\left(x+dy\right)\right)^2 \csc h\left(K\left(x+dy\right)\right)^4 \sec h\left(K\left(x+dy\right)\right)^4}{4}} \tag{11}$$

$$+4m^2 \arccos\left[\frac{\coth\left(K\left(x+dy\right)\right)^2\left(1+\tanh\left(K\left(x+dy\right)\right)^4\right)}{2}\right]\cosh\left(4K\left(x+dy\right)\right)$$

$$x\sqrt{\frac{-\cosh\left(2K\left(x+dy\right)\right)^2 \csc h\left(K\left(x+dy\right)\right)^4 \sec h\left(K\left(x+dy\right)\right)^4}{4}}$$

$$-m^2 \arccos\left[\frac{\coth\left(K\left(x+dy\right)\right)^2\left(1+\tanh\left(K\left(x+dy\right)\right)^4\right)}{2}\right]\cosh\left(8K\left(x+dy\right)\right)$$

$$x\sqrt{\frac{-\cosh\left(2K\left(x+dy\right)\right)^2 \csc h\left(K\left(x+dy\right)\right)^4 \sec h\left(K\left(x+dy\right)\right)^4}{4}},$$

…

and so on, in this manner the rest of components of the homotopy series can be obtained. The solution $u(x,y,t)$ in a series form and the series can be written in a closed form solution by

$$u\left(x,y,t\right) = \arccos\left[\frac{\coth^2\left(K\left(ct+x+dy\right)\right)\left(1+\tanh^4\left(K\left(ct+x+dy\right)\right)\right)}{2}\right] \tag{12}$$

For second example, we consider the tdSG Eq. (1) with the initial conditions

$$u\left(x,y,0\right) = \arccos\left[\frac{(1+\coth\left(K\left(x+dy\right)\right)^4)\tanh\left(K\left(x+dy\right)\right)^2}{2}\right],$$

$$u_t(x,y,0) = \frac{cK\cosh\left(2K\left(x+dy\right)\right)\csc h\left(K\left(x+dy\right)\right)^3 \sec h\left(K\left(x+dy\right)\right)^3}{\sqrt{1-\frac{(1+\coth\left(K\left(x+dy\right)\right)^4)^2 \tan h\left(K\left(x+dy\right)\right)^4}{4}}} \tag{13}$$

where a_1,a_3 and a_5 are constants and $c = \sqrt{a_1 - \frac{3a_3^2}{16a_5}}, d = \frac{2\left(c^2 - a_1\right)}{a_3}, K = \sqrt{\frac{c^2 - a_1}{c^2}}$
Similarly, we obtain

$$u_0 = 0, \tag{14}$$

$$u_1 = \arccos\left[\frac{(1+\coth(K(x+dy))^4)\tanh(K(x+dy))^2}{2}\right] + \frac{cK\cosh(2K(x+dy))csh(K(x+dy))^3 \sec h(K(x+dy))^3}{\sqrt{1-\frac{(1+\coth(K(x+dy))^4)\tanh(K(x+dy))^4}{4}}}. \tag{15}$$

$\ldots$

and so on, in this manner the rest of components of the homotopy series can be obtained. The solution $u(x,y,t)$ in a series form and the series can be written in a closed form solution by

$$u(x,y,t) = \arccos\left[\frac{(1+\coth(K(ct+x+dy))^4)\tanh(K(ct+x+dy))^2}{2}\right] \tag{16}$$

Table 1: The absolute value ($|u(0.5,y,t)-\phi_5(0.5,y,t)|$) of the numerical results when $m = 5$, $\lambda = 1$, $K = 5$ and $d = \frac{1}{2}\sqrt{-4+4\lambda^2+\frac{m^2}{K^2}}$ for the solution of Eq. (1) for initial conditions (4).

(x_i,t_i) (0.01, 0.01)	(0.02, 0.02)	(0.03, 0.03)	(0.04, 0.04)	(0.05, 0.05)
c=0.5				
0.00009	0,00038	0.00086	0.00152	0.00235
0.00009	0.00038	0.00085	0.00150	0.00233
0.00009	0,00038	0,00084	0,00149	0,00230
0.00009	0.00037	0.00083	0.00147	0.00228
0.00009	0.00037	0.00082	0.00146	0.00226
c=0.1				
$2x10^{-14}$	$3x10^{-13}$	$3x10^{-12}$	10^{-11}	$5x10^{-11}$
$4x10^{-15}$	$3x10^{-13}$	$3x10^{-12}$	10^{-11}	$5x10^{-11}$
$6x10^{-15}$	$3x10^{-13}$	$3x10^{-12}$	10^{-11}	$5x10^{-11}$
$2x10^{-15}$	$2x10^{-13}$	$3\ x10^{-12}$	10^{-11}	$4x10^{-11}$
$2x10^{-14}$	$2x10^{-13}$	$3x10^{-12}$	10^{-11}	$4x10^{-11}$

Table 2: The absolute value ($|u(0.5,y,t)-\phi_5(0.5,y,t)|$) of the numerical results when $m = 5$, $\lambda = 1$, $K = 5$ and $d = \frac{1}{2}\sqrt{-4+4\lambda^2+\frac{m^2}{K^2}}$ for the solution of Eq. (1) for initial conditions (13).

(x_i,t_i) (0.01, 0.01)	(0.02, 0.02)	(0.03, 0.03)	(0.04, 0.04)	(0.05, 0.05)
c=0.5				
0.00009	0.00038	0.00086	0.00152	0.00235
0.00009	0.00038	0.00085	0.00150	0.00233
0.00009	0.00038	0.00084	0.00149	0.00230
0.00009	0.00037	0.00083	0.00147	0.00228
0.00009	0.00037	0.00082	0.00146	0.00226
c=0.1				
10^{-14}	$3x10^{-13}$	$3x10^{-12}$	10^{-11}	$5x10^{-11}$
$5x10^{-15}$	$3x10^{-13}$	$3x10^{-12}$	10^{-11}	$5x10^{-11}$
$2x10^{-15}$	$3x10^{-13}$	$3x10^{-12}$	10^{-11}	$5x10^{-11}$
$3x10^{-15}$	$3x10^{-13}$	$3x10^{-12}$	10^{-11}	$4x10^{-11}$
$2x10^{-14}$	$3x10^{-13}$	$3x10^{-12}$	10^{-11}	$4x10^{-11}$

RESULTS AND DISCUSSION

We try to obtain numerical solutions of two dimensional sine-Gordon equations. We evaluate the approximate solution using the 5-term approximation (ϕ_5). In the Tables 1 and 2, the numerical results obtained for approximate solution of Eq. (1) by using the homotopy perturbation method. We achieved a very good approximation by using 5 terms only of the homotopy perturbation series. We can show that even using few terms of series, the overall results getting very close to exact solution, errors can be made smaller by adding new terms of the homotopy series.

CONCLUSION

In this paper, we apply HPM for solving two dimensional sine-Gordon equation. We obtained a very good approximation by using 5 terms of the homotopy perturbation series. The numerical results show that proposed method is very accurate. One concludes that the suggested technique is very simple and straightforward. Also, this approach does not require any discretization, linearization or small perturbations and therefore is capable of greatly reducing the size of calculations while still maintaining high accuracy of the numerical solution.

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