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# A New Linear Differential Operator and its Application

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**Abstract:** Using the convolution technique we introduce a new linear differential operator on the class *A* of analytic functions in the open unit disk *E*. By using this operator we define some new classes of analytic functions and study some basic properties of these classes such as, rate of growth of coefficients, inclusion result and radius problem. We also show that these classes are closed under convolution with convex function.

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## INTRODUCTION

Let A denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E \tag{1.1}$$

which are analytic in the open unit disk E = {z: |z|<1}. Let S, $S^*$ and C denotes the classes of all those functions in A which are univalent, starlike, convex respectively.

For any two analytic functions

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n$$

and

$$f_2(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in E$$

the convolution (Hadamard product) of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = \sum_{n=1}^{\infty} a_n b_n z^n, \quad z \in E$$

Using the concept of convolution, we define a linear operator $D_k^\delta : A \to A$ by

$$D_k^\delta f(z) = \frac{z(1+z)^{\left(\frac{k}{4}-\frac{1}{2}\right)(\delta+1)}}{(1-z)^{\left(\frac{k}{4}+\frac{1}{2}\right)(\delta+1)}} * f(z), \quad \text{for } \delta > -1, k \geq 2$$

$$= z + \sum_{n=2}^{\infty} \varphi_k(n,\delta) a_n z^n$$

with

$$\varphi_k(n,\delta) = \sum_{j=1}^{n} \frac{\left(\left(\frac{k}{4}+\frac{1}{2}\right)(1+\delta)\right)_{j-1}}{(j-1)!} \frac{\prod_{m=0}^{n-j-1}\left(\left(\frac{k}{4}-\frac{1}{2}\right)(1+\delta)-m\right)}{(n-j)!}$$

where $(\rho)_m$ is a Pochhammer symbol defined as

$$(\rho)_m = \begin{cases} 1, m = 0 \\ \rho(\rho+1)(\rho+2)\ldots(\rho+m-1), m \in \mathbb{N} \end{cases}$$

We see that, when k = 2, the above linear operator $D_k^\delta$ reduces to the linear operator $D^\delta$, defined by Ruscheweyh [5]. For application of Ruscheweyh operator [2-4].

Using the operator $D_k^\delta$, we define the following new class of analytic functions.

**Definition 1.1:** Let β≥0, k≥2 and α>-1. Then *f*(z)∈R(k,α,β), if and only if,

$$\left(D_k^\alpha f(z)\right)' + \beta z\left(D_k^\alpha f(z)\right)'' \in P, z \in E$$

## PRELIMINARY RESULTS

**Lemma 2.1:** If p(z) is analytic in E, p(0) = 1 and Re p(z)>1/2, z∈E, then for any function F(z), analytic in E, the function p(z)* *f*(z) takes values in the convex hull of the image of E under F(z) [6].

**Lemma 2.2:** Let β≥0, D(z)∈$S^*$ and N(z) be analytic in E with N(0) = D(0) = 0, N(0) = D′(0) = 1. Let for z∈E [1],

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$$(1-\beta)\frac{N(z)}{D(z)}+\beta\frac{N'(z)}{D'(z)}\in P.$$

Then

$$\frac{N(z)}{D(z)}\in P$$

## MAIN RESULTS

**Theorem 3.1:** The class $R(k,\alpha,\beta)$ is a convex set.

**Proof:** Let $f(z)$, $g(z)\in R(k,\alpha,\beta)$. Then

$$\left(D_k^\alpha f(z)\right)'+\beta z\left(D_k^\alpha f(z)\right)''=h_1(z) \tag{3.1}$$

$$\left(D_k^\alpha g(z)\right)'+\beta z\left(D_k^\alpha g(z)\right)''=h_2(z) \tag{3.2}$$

where $h_1(z)$, $h_2(z)\in P$.
Let

$$F(z)=(1-\lambda)f(z)+\lambda g(z),\lambda\in(0,1)$$

then

$$D_k^\alpha F(z)=(1-\lambda)D_k^\alpha f(z)+\lambda D_k^\alpha g(z) \tag{3.3}$$

From (3.1), (3.2) and (3.3), we have

$$\left(D_k^\alpha F(z)\right)'+\beta z\left(D_k^\alpha F(z)\right)''=(1-\lambda)h_1(z)+\lambda h_2(z)\in P$$

since P is a convex set. This implies that $F(z)\in R(k,\alpha,\beta)$ and hence $R(k,\alpha,\beta)$ is a convex set.

**Theorem 3.2:** Let $k\geq 2$, $\alpha>-1$ and $\beta\geq 0$. Then $R(k,\alpha,\beta)\subset R(k,\alpha,0)$.

**Proof:** Let $N(z)=z\left(D_k^\alpha f(z)\right)'$ and $D(z)=z$. Then $D(z)=z\in S^*$ and

$$N(0)=D(0)=0,N'(0)=D'(0)=1$$

Now consider

$$(1-\beta)\frac{N(z)}{D(z)}+\beta\frac{N'(z)}{D'(z)}=\left(D_k^\alpha f(z)\right)'+\beta z\left(D_k^\alpha f(z)\right)''\in P$$

since $f(z)\in R(k,\alpha,\beta)$. Therefore by using Lemma 2.2, we have $\frac{N(z)}{D(z)}\in P$, that is, $\left(D_k^\alpha f(z)\right)'\in P$. This implies $f(z)\in R(k,\alpha,0)$.

**Theorem 3.3:** Let $f(z)\in R(k,\alpha,0)$. Then $f(z)\in R(k,\alpha,\beta)$ for $|z|<r_\beta$, where

$$r_\beta=\frac{1}{2\beta+\sqrt{4\beta^2-2\beta+1}} \tag{3.4}$$

**Proof:** Let

$$\Phi_\beta(z)=(1-\beta)\left(D_k^\alpha f(z)\right)'+\beta\left(z\left(D_k^\alpha f(z)\right)'\right)'.$$

Then

$$\Phi_\beta(z)=\frac{q(z)}{z}*\left(D_k^\alpha f(z)\right)'$$

where

$$q(z)=(1-\beta)\frac{z}{1-z}+\beta\frac{z}{(1-z)^2}$$

As $q(z)$ is convex for $|z|<r_\beta$ and consequently, for $|z|<r_\beta$, $\mathrm{Re}\frac{q(z)}{z}>\frac{1}{2}$. Then, by using Lemma 2.1, $\Phi_\beta(z)$ takes values in the convex hull of $F(E)$. This implies that $f(z)\in R(k,\alpha,\beta)$ for $|z|<r_\beta$ as given in (3.4).

**Theorem 3.4:** Let $f(z)\in R(k,\alpha,\beta)$ and $\varphi(z)\in C$. Then

$$f(z)*\varphi(z)\in R(k,\alpha,\beta)$$

**Proof:** Let $f(z)\in R(k,\alpha,\beta)$. Then there exists $h(z)\in P$ such that

$$\left(D_k^\alpha f(z)\right)'+\beta z\left(D_k^\alpha f(z)\right)''=h(z) \tag{3.5}$$

We denote

$$G(z)=f(z)*\varphi(z).$$

Then

$$D_k^\alpha G(z)=\varphi(z)*D_k^\alpha f(z),$$

and therefore, by using (3.5), we have

$$\begin{aligned}\left(D_k^\alpha G(z)\right)'+\beta z\left(D_k^\alpha G(z)\right)''&=\frac{\varphi(z)}{z}*\left\{\left(D_k^\alpha f(z)\right)'+\beta z\left(D_k^\alpha f(z)\right)''\right\}\\&=\frac{\varphi(z)}{z}*h(z)\end{aligned}$$

As $\varphi(z)$ is a convex then by using Lemma 2.1, we obtain the required result.

**Application of Theorem 3.4:** The class $R(k,\alpha,\beta)$ is invariant under the following operators.

(i) $f_1(z) = \int_0^z \frac{f(\xi)}{\xi} d\xi$

(ii) $f_2(z) = \frac{2}{z} \int_0^z f(\xi) d\xi$

(iii) $f_3(z) = \int_0^z \frac{f(\xi) - f(x\xi)}{\xi - x\xi} d\xi \quad |x| \le 1, x \ne 1$

(iv) $f_4(z) = \frac{1+c}{z^c} \int_0^z \xi^{c-1} f(\xi) d\xi, \quad \mathrm{Re}\, c > 0$

Since

$$f_i(z) = f(z) * \varphi_i(z)$$

where

$$\varphi_1(z) = -\log(1-z)$$

$$\varphi_2(z) = \frac{-2[z - \log(1-z)]}{z}$$

$$\varphi_3(z) = \frac{1}{1-x} \log\left(\frac{1-xz}{1-z}\right), \ |x| \le 1, x \ne 1$$

$$\varphi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \ \mathrm{Re}\, c > 0$$

and each $\varphi_i(z)$, $1 \le i \le 4$, is convex.

**Theorem 3.5:** Let $f(z) \in R(k,\alpha,\beta)$. Then

$$|a_n| \le \frac{2}{n[1+\beta(n-1)]|\Psi_k(n,\alpha)|}, \text{ for all } n \ge 2$$

where $\Psi_k(n,\alpha)$ is given by (1.3).

**Proof:** Let $f(z) \in R(k,\alpha,\beta)$. Then

$$\left(D_k^{\alpha} f(z)\right)' + \beta z \left(D_k^{\alpha} f(z)\right)'' = h(z), \quad h(z) \in P \tag{3.6}$$

Let

$$h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \ z \in E \tag{3.7}$$

Therefore, by using (1.2), (3.6) and (3.7), we easily have

$$|a_n| \le \frac{2}{n[1+\beta(n-1)]|\Psi_k(n,\alpha)|}, \text{ for all } n \ge 2$$

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