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Sum Formulas for Fibonacci Numbers

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Abstract: We give an elementary deduction of the sum formulas of the squares of terms obtained by Soykan for Fibonacci numbers.

Key words: Fibonacci numbers · d'Ocagne's identity · Soykan's formulas

INTRODUCTION

Soykan [1] obtained the following relations:

$$\sum_{k=1}^{n} k\ F_k^2 = \frac{1}{2}[1+(3+2n)(F_{n+2}-F_{n+1})F_{n+1}-F_{n+2}^2], \tag{1}$$

$$\sum_{k=1}^{n} k\ F_{k+1}F_k = \frac{1}{2}[(1+2n)F_{k+2}^2+(3+2n)\ F_{n+1}^2-(5+2n)\ F_{n+2}\ F_{n+1}+1], \tag{2}$$

involving the Fibonacci numbers [2-10] defined via the recurrence relation:

$$F_{n+1}=F_n+F_{n-1}, \qquad F_0=0, \qquad F_0=1, \qquad n\geq 1, \tag{3}$$

thus $\{F_n\} = \{0, 1, 1, 2, 3, 5, 8, 13, ...\}$, which can be generated with the expression [11]:

$$F_n=\frac{a^n-b^n}{a-b}, \qquad a=\frac{1+\sqrt{5}}{2}, \qquad b=\frac{1-\sqrt{5}}{2}, \tag{4}$$

that is:

$$a-b=\sqrt{5}, \quad a+b=1, \quad ab=-1, \quad 1-ab=2, \quad a^2-1=a, \quad b^2-1=b. \tag{5}$$

On the other hand, from the geometric series of Gauss:

$$\sum_{k=1}^{m} k\ \ c^k = \frac{c}{(1-c)^2}\ [1-(m+1)\ c^m+m\ c^{m+1}]. \tag{6}$$

From (4), $(a-b)^2F_k^2=a^{2k}-2(ab)^k+b^{2k}$, then (6) allows obtain the relation:

$$(a-b)^2\sum_{k=1}^{n} k\ F_k^2 = \frac{a^2}{(1-a^2)^2}[1-(n+1)a^{2n}+n\ a^{2n+2}]\ +\frac{b^2}{(1-b^2)^2}\ [1-(n+1)\ b^{2n}+n\ b^{2n+2} - \frac{2ab}{(1-ab)^2}\ [1-(n+1)\ (ab)^n\ +n(ab)^{n+1}]$$

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where can be applied the values (5) to deduce the property:

$$\sum_{k=1}^{n} k\, F_k^2 = [n\, F_{n+2} - (n+1)\, F_n]\, F_n + \frac{1-(-1)^n}{2}, \tag{7}$$

which is alternative to (1); in the deduction of (7) it was used the d'Ocagne's identity [11]:

$$F_{n+1}^2 = F_n\, F_{n+2} + (-1)^n. \tag{8}$$

Similarly:

$$(a-b)^2\, F_{k+2}\, F_k = a^{2k+1} + b^{2k+1} - (a+b)(ab)^k,$$

hence from (6):

$$(a-b)^2 \sum_{k=1}^{n} k\, F_{k+1}\, F_k = \frac{a^2}{(1-a^2)^2}\,[1-(n+1)a^{2n} + n\, a^{2n+2}] + \frac{b^2}{(1-b^2)^2}\,[1-(n+1)\, b^{2n} + n\, b^{2n+2} - \frac{ab(a+b)}{(1-ab)^2}\,[1-(n+1)\,(ab)^n + n(ab)^{n+1}],$$

which with the values (5) takes the final form:

$$\sum_{k=1}^{n} k\, F_{k+1}\, F_k = (n\, F_{n+2} - F_{n+1})\, F_n + \frac{1-(-1)^n}{4} + \frac{n(-1)^n}{2}, \tag{9}$$

as an alternative for (2).

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