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Comparison of Numerical Methods for Approximating Solution of the Quadratic Riccati Differential Equation

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Abstract: In this paper, the combined Newton's interpolation and Aitken's method as a hybrid technique is applied to solve nonlinear Riccati differential equation. In general we find that, The results are not effective at the points of solution where the step of partition of the interval of the interval containing the variable x are large, this result give us the solution near the result of variational iteration method (VIM) studied by Belal. B. While we found for a small step of partition that, the results are near the exact solutions and near the multistage variational method (MVIM) studied by Belal. B.

Key words: Quadratic Riccati equation・Newton's interpolation and Aitken's method・Variational iteration method・Multistage variational iteration method

INTRODUCTION

The Riccati differential equation is named after the Italian nobleman Count Jacopo Francesco Riccati (1676-1754) [1, 2]. There has been greater attempt to solving differential equations by numeric-analytic methods. Most of authors treated a numeric-analytic method for approximating quadratic Riccati differential equation (RDE), variational iteration method (VIM) studied by Belal. B [2], Multistage variational method (MVIM) [2]. Numerical Laplace transform method is applied to approximate the solution of nonlinear (quadratic) Riccati differential equations mingled with Adomian decomposition method. A new technique is given by Vinod M and Dimple R [3], by reintroducing the unknown function in Adomian polynomial with that of well-known Newton-Raphson formula. Besides important engineering science applications that today are considered classical, such as stochastic realization theory, optimal control, robust stabilization and network synthesis, the newer applications include such areas as ¯nancial mathematics [4, 5]. The solution of this equation can be reached using classical numerical methods such as the forward Euler method and Runge-Kutta method. An unconditionally stable scheme was presented by Dubois F and Saidi A [6]. El-Tawil, *et al*. [7] presented (MVIM) for quadratic Riccati Equation, the usage of Adomian decomposition method (ADM) to solve the nonlinear Riccati in an analytic form. Tan and Abbasbandy [8] employed the analytic technique called Homotopy Analysis Method (HAM) to solve aquadratic Riccati equation. GengF [9] presented a modi¯edvari-ational iteration method to solve quadratic Riccati equation. The variational iteration method (VIM) is a simple and yet powerful method for solving a wide class of nonlinear problems, first envisioned by He [10]. Belal B. gave a modified version of VIM, which is called multi- stage variational iteration method (MVIM) and he presented a comparative solutions with (VIM) and exact solution. The investigated (VIM) and (MVIM) accuracy for a longer time frame to show its reliability to Riccati equation. In this paper hybrid Newton's interpolation and Aitken's method technique is applied to solve nonlinear Riccati differential equation. We consider the following nonlinear, Riccati differential equation (RDE) of the form [3]:

$$\frac{dy}{dx} = q(x)y + r(x)y^2 + p(x), y(0) = a \tag{1}$$

Combined Newton's Interpolation and Lagrange Method [11]: This study combine both Newton's interpolation method and Lagrange method. It used newton's interpolation method to find the second two terms then use the three values for y to form a quadratic equation using Lagrange interpolation method as follows;

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Newton’s interpolation method [11].

$$f_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots a_2(x - x_{n-1}) \tag{2}$$

where,

$$a_0 = y_0, a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}, a_2 = \frac{\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)} \tag{3}$$

etc

Lagrang interpolation method [11].

$$y_n = \frac{(x - x_1) - (x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0) - (x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0) - (x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2 \tag{4}$$

Description of the Hybrid Method: This method will combine both Newton’s interpolation method and Aitken method. It used newton’s interpolation method to find the second two terms then use the three values for y to form a linear or quadratic equations using Aitken interpolation method as follows;

$$f_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots a_2(x - x_{n-1}) \tag{5}$$

where,

$$a_0 = y_0, a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}, a_2 = \frac{\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)} \tag{6}$$

etc

$$y_1 = a_0 + a_1(x - x_0) \tag{7}$$

$$y_2 = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \tag{8}$$

Note: We can use Newton's Forward Interpolation Formula instead of Newton's divided Interpolation method in (2.1).

Aitken Interpolation Method:

$$P_{o,k}(x) = \frac{1}{x_k - x_o} \begin{vmatrix} y_o & x_o - x \\ y_k & x_k - x \end{vmatrix} \tag{9}$$

$$P_{o,1,2}(x) = \frac{1}{x_2 - x_1} \begin{vmatrix} P_{o,1}(x) & x_1 - x \\ P_{o,2}(x) & x_2 - x \end{vmatrix} \tag{10}$$

$$y_n = P_{o,1,2,\ldots,n}(x) = \frac{1}{x_n - x_{n-1}} \begin{vmatrix} P_{o,1,\ldots,(n-1)}(x) & x_{n-1} - x \\ P_{o,1,\ldots,(n-2),n}(x) & x_n - x \end{vmatrix} \tag{11}$$

Example: In this section, we will check the effectiveness of the present technique.

Solve $\frac{dy}{dx} = 1 - y^2(x) + 2y(x), \quad y(0) = 0,\ x \in [0, 20]$

The exact solution is [2]

$$y(x) = 1 + \sqrt{2} tanh(\sqrt{2x} + \frac{1}{2}\log\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right))$$

By taking the step h = 2

First by using Newton's interpolation, we have,

$$a_0 = 0 = y_0$$
$$a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = [\frac{dy}{dx}]_{0,0} = 1$$
$$y_1 = 0 + 1.(2 - 0) = 2$$
$$a_2 = \frac{[\frac{dy}{dx}]_{2,2} - [\frac{dy}{dx}]_{0,0}}{2 - 0} = 0$$
$$y_2 = 0 + 1(2 - 0) = 2$$

Now, forming linear and quadratic using Aitken Method

$$P_{0,1}(x) = x$$
$$P_{0,2}(x) = 0.5x$$
$$P_{0,1,2}(x) = -0.25x^2 + 1.5x$$

Hence, we can take the approximation solution of quadratic using Aitken Method, we find the solution given by Table 1. It is very far from the exact solution and also far from the solution given by the (VIM) method [2].

To solve this problem, for some points only, Belal B. [1] took the step h=0.2 in VIM method. by the Hybrid method taking h small, h=0.2 we can see the solution.

Table 1: Solution of $\frac{dy}{dx} = 1 - y^2(x) + 2y(x),\ y(0) = 0, x \in [0,20]$ In comparison with Exact solution, (MVIT) method and (VIM) method for h=2

x	Exact solution	2-Iterate MVIM	3-Iterate VIM	Combined Newton's Interpolation and Aitken *for h=2*
2	2.3577716530	2.3592420980	-286352.73097900	2.0000000000
4	2.4140123820	2.4140330560	-9.0657980428E19	2.0000000000
6	2.4142128590	2.4142130890	-7.3332282199E33	-3.000000000
8	2.4142135600	2.4142136400	-5.7930892793E47	-4.000000000
10	2.4142135620	2.4142136400	-4.5744317440E61	-10.00000000
12	2.4142135620	2.4142136400	-3.6121072512E75	-18.00000000
14	2.4142135620	2.4142136560	-2.8522268182E89	-28.00000000
16	2.4142135620	2.4142136940	-2.2522027269E103	-40.00000000
18	2.4142135620	2.4142136830	-1.7784059425E117	-54.00000000

Table 2: Solution of $\frac{dy}{dx} = 1 - y^2(x) + 2y(x),\ y(0) = 0, x \in [0,2]$

x	Exact solution	2-Iterate MVIM	3-Iterate VIM	Combined Newton's Interpolation and Aitken *for h=0.2*	Combined Newton's Interpolation and Aitken *for h=0.01*
0.2	0.2419767992	0.2396149017	0.2419778327	0.200000000	0.237810000
0.4	0.5678121656	0.5626231618	0.5678455132	0.472000000	0.555220000
0.6	0.9535662155	0.9468409011	0.9536660329	0.816000000	0.952230000
0.8	1.3463636550	1.3405640980	1.3463791062	1.232000000	1.428840000
1.0	1.6894983900	1.6863821450	1.6860271032	1.720000000	1.985050000
1.2	1.9513601180	1.9509491870	1.9150510260	2.280000000	2.620860000
1.4	2.1313266100	2.1325827440	2.1791315021	2.912000000	3.336270000
1.6	2.2462859590	2.2481414290	-50.98229780	3.616000000	4.131280000
1.8	2.3163247370	2.3181237490	-5338.782860	4.392000000	5.005890000
2.0	2.3577716530	2.3592420980	-286352.7325	5.240000000	7.940200000

First by using Newton's interpolation, we have,

$$a_0 = 0 = y_0$$
$$a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = [\frac{dy}{dx}]_{0,0} = 1$$
$$y_1 = 0 + 1(0.2 - 0) = 0.2$$
$$a_2 = \frac{[\frac{dy}{dx}]_{0.2,0.2} - [\frac{dy}{dx}]_{0,0}}{0.4 - 0} = 0.9$$
$$y_2 = 0 + 1(0.4 - 0) + 0.9(0.4 - 0)(0.4 - 0.2) = 0.472$$

Now, forming linear and quadratic using Hybrid Method.

$$P_{0,1}(x) = x$$
$$P_{0,2}(x) = 0.18x$$
$$P_{0,1,2}(x) = -0.9x^2 + 0.82x$$

Hence, we can take the approximation solution of quadratic using Hybrid Method, we find the solution given by Table 2. It is also far from the exact solution and also far from the solution given by the (VIM) method [2]. Finally by taking the step h = 0.01, we have the following linear and quadratic using Hybrid Method.

$$a_0 = 0.5 = y_0$$
$$a_1 = \frac{f(x_1 - x_0)}{(x_1 - x_0)} = [\frac{dy}{dx}]_{0,0.5} = 0.5$$
$$y_1 = 0.5 + 0.5(0.01 - 0) = 0.505$$
$$a_2 = a_2 = \frac{\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)} = \frac{[\frac{dy}{dx}]_{0.01,0.505} - [\frac{dy}{dx}]_{0,0.5}}{0.02 - 0} = -0.25$$
$$y_2 = 0.5 + 0.5(0.02 - 0) - 0.25(0.02 - 0)(0.02 - 0.01) = 0.50995$$

Now, forming linear and quadratic using Aitken Method.

$$P_{0,1}(x) = x$$
$$P_{0,2}(x) = 1.00995x$$
$$P_{0,1,2}(x) = -0.995x^2 + 0.99005x$$

Now let's use this approximation to get the solution for x=0.2, 0, 4, 0.6, ….

We find the results in Table 2. Which shows us a significant improvement in the solution, but not in all points, only in the first three points and therefore it can be considered useful to get the solution in these three points only.

CONCLUSIONS

In this work, we have been solve the Riccati nonlinear first order differential equation by the combined Newton's interpolation and Aitken's method, we compare the result for one example with exact solution, variational iteration method (VIM) and multistage variational method (MVIM). We found that using this combined method is not generally effective, not stable and this method can be improved by minimizing the step h for the solution interval and obtaining the approximation relationship, then using it in a limited number of first points of the solution interval.

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