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An Identity for Invariant Sequences

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Abstract: We exhibit an elementary deduction of an identity of Sun for invariant sequences.

Key words: Invariant sequences - Binomial transformation - Bernoulli numbers

INTRODUCTION

We say that $\{a_n\}$ is an invariant sequence if:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k = a_n , \quad n \geq 0, \tag{1}$$

Then Sun [1] showed, for this type of sequences, the following identity:

$$\sum_{k=0}^{n} \binom{n}{k} f(k)\, a_{n-k} = \sum_{r=0}^{n} f(r) \sum_{k=r}^{n} \binom{n}{k} \binom{k}{r} (-1)^{n-k} a_{n-k} , \tag{2}$$

where f is an arbitrary function. Here we give an elementary proof of (2), in fact, the property (1) means, in the Rainville's notation:

$$(1-a)^n = a^n, \quad a^k \equiv a_k , \tag{3}$$

Then:

$$\sum_{k=0}^{n} \binom{n}{k} f(k)\, a_{n-k} = (f+a)^n = [(f+1)-(1-a)]^n = \sum_{q=0}^{n} \binom{n}{q} (f+1)^{n-q} (-1)^q (1-a)^q ,$$

$$\overset{(3)}{=} \sum_{q=0}^{n} \binom{n}{q} (-1)^q a_q \sum_{r=0}^{n-q} \binom{n-q}{r} f^r = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_{n-k} \sum_{r=0}^{k} \binom{k}{r} f(r),$$

Which is equivalent to (2), q.e.d.

We know [1] that $\{(-1)^n B_n\}$ is an invariant sequence involving the Bernoulli numbers [2-4], thus (2) implies the relation:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k)\, B_{n-k} = \sum_{r=0}^{n} f(r) \sum_{k=r}^{n} \binom{n}{k} \binom{k}{r} B_{n-k} , \tag{4}$$

But we have the expression [5]:

$$\sum_{k=r}^{n} \binom{n}{k} \binom{k}{r} B_{n-k} = \binom{n}{r} [\delta_{n-r,1} + B_{n-r}], \tag{5}$$

Hence (4) takes the form:

$$\sum_{k=0}^{n} \binom{n}{k} [(-1)^{n-k} - 1]\, f(k)\, B_{n-k} = n\, f(n-1), \quad n \geq 0, \tag{6}$$

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That is:

$$\sum_{k=0}^{n-2} \binom{n}{k} [(-1)^{n-k} - 1] f(k) B_{n-k} = 0, \quad n \geq 2 . \tag{7}$$

For example, the identity (2) can be applied to the following invariant sequences:

$$\left\{\frac{1}{2^n}\right\}, \left\{\frac{1}{\binom{n+2m-1}{m}}\right\}, m \geq 1, \{n F_{n-1}\}, \{L_n\}, \left\{\frac{T_n(x)}{(2x)^n}\right\}, \left\{\frac{(-1)^{n+1}(2^{n+1}-1)}{n+1} B_{n+1}\right\}, \tag{8}$$

$$\left\{(-1)^n \int_0^{2m-1} \binom{x}{n+2m} dx\right\}, m \geq 0, \quad \left\{\frac{\binom{2n}{n}}{2^{2n}}\right\}, \quad \left\{\frac{\binom{\frac{x}{2}}{n}}{\binom{x}{n}}\right\}, x \neq 0, 1, 2, \ldots$$

Involving Che by shev polynomials, Fibonacci and Lucas numbers.

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