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Bernoulli and Euler Numbers in Terms of Stirling Numbers of the Second Kind

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Abstract: We give an elementary deduction of an identity obtained by Jha for the Bernoulli numbers involving the Stirling numbers of the second kind $S_n^{[k]}$. Besides, we employ a relation of Guo-Qi to write the Euler numbers in terms of the $S_m^{[j]}$.

Key words: Stirling numbers - Guo-Qi's identity - Bernoulli numbers - Euler polynomials

INTRODUCTION

We have the following Jha's relation [1-3]:

$$B_n = \sum_{k=1}^{n} (-1)^{n+k-1} \frac{(k-1)!}{k+1} S_n^{[k]} , \tag{1}$$

Involving the Bernoulli and Stirling numbers of the second kind [4-6]; now we shall give a simple proof of (1), in fact:

$$\sum_{k=j}^{n} \frac{1}{k\,(k+1)} \binom{k}{j} = \sum_{k=j}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \binom{k}{j} = \sum_{k=j}^{n} \frac{1}{k} \binom{k}{j} - \sum_{k=j}^{n} \frac{1}{k+1} \binom{k}{j}, \tag{2}$$

But we know the hockey stick identity [7-10]:

$$\sum_{k=j}^{n} \frac{1}{k} \binom{k}{j} = \frac{1}{j} \binom{n}{j}, \tag{3}$$

and the expression [6]:

$$B_n = \sum_{j=1}^{n} (-1)^j \, j^n \sum_{k=j}^{n} \frac{1}{k+1} \binom{k}{j}, \tag{4}$$

then from (2), (3) and (4):

$$\begin{aligned} B_n &= \sum_{j=1}^{n} (-1)^{j-1} j^n \sum_{k=j}^{n} \frac{1}{k\,(k+1)} \binom{k}{j} + \sum_{j=1}^{n} (-1)^j \binom{n}{j} j^{n-1} , \\ &= -\sum_{k=1}^{n} \frac{1}{k\,(k+1)} \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^n - \delta_{1n} = \sum_{k=1}^{n} (-1)^{k-1} \frac{(k-1)!}{k+1} S_n^{[k]} - \delta_{1n} , \ n \geq 1, \end{aligned} \tag{5}$$

Which is equivalent to (1), q.e.d. In (5) we used the known relation [5, 6]:

$$S_n^{[k]} = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n. \tag{6}$$

On the other hand, we have the following property [4, 11]:

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k} , \tag{7}$$

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Involving the numbers and polynomials of Euler, and also the formula obtained by Guo-Qi [12]:

$$E_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left[\sum_{r=1}^{n-k+1} \frac{(-1)^{r-1}(r-1)!}{2^{r-1}} S_{n-k+1}^{[r]} \right] x^k, \tag{8}$$

Then in (7) and (8) we can match the coefficients of x^k to obtain:

$$(E-1)^n \equiv \sum_{k=0}^{n} (-1)^k \binom{n}{k} E_k = (-2)^{n+1} \sum_{r=1}^{n+1} \frac{(-1)^r (r-1)!}{2^r} S_{n+1}^{[r]}, \tag{9}$$

where it was applied the Rainville's notation $E_r \equiv E^r$. We know the expressions [11]:

$$E_n(x) + \sum_{k=0}^{n} \binom{n}{k} E_k(x) = 2\,x^n, \qquad E_n = 2^n E_n\left(\tfrac{1}{2}\right), \tag{10}$$

Therefore:

$$(E+2)^n \equiv \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} E_k = 2 - E_n, \tag{11}$$

Hence from (11):

$$(E+2)^n = (E-1+3)^n = \sum_{q=0}^{n} \binom{n}{q} 3^{n-q} (E-1)^q,$$

where we can employ (9) to deduce the identity:

$$E_n = 2\left[1 + 3^n \sum_{k=0}^{n} \binom{n}{k} \left(-\tfrac{2}{3}\right)^k \sum_{r=1}^{k+1} \frac{(-1)^r (r-1)!}{2^r} S_{k+1}^{[r]}\right], \quad n \geq 0, \tag{12}$$

That is, the Euler numbers in terms of Stirling numbers of the second kind.

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