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On Some Series Representations for Riemann Zeta (3)

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Abstract: We show that the series representations of Euler and Zhang-Williams for ζ(3) imply the representations of Chen-Srivastava and Scheufens for this value of the Riemann zeta function.

Key words: Riemann zeta function - Series representations - Scheufens expansion

INTRODUCTION

In the literature are the following series representations for Riemann zeta (3):

$$A \equiv \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+2)(2k+3)} \cdot \frac{\zeta(2k)}{4^k} = -\frac{5}{8\pi^2}\zeta(3), \quad (1)$$

$$B \equiv \sum_{k=0}^{\infty} \frac{2k-1}{(2k+1)(2k+2)(2k+3)} \cdot \frac{\zeta(2k)}{4^k} = \frac{3}{4\pi^2}\zeta(3), \quad (2)$$

Deduced by Chen-Srivastava [1-3] and Scheufens [4], respectively. On the other hand, we have the relations:

$$C \equiv \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+2)} \cdot \frac{\zeta(2k)}{4^k} = -\frac{7}{4\pi^2}\zeta(3), \quad (3)$$

$$D \equiv \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)} \cdot \frac{\zeta(2k)}{4^k} = -\frac{9}{8\pi^2}\zeta(3), \quad (4)$$

Obtained by Euler [2-10] and Zhang-Williams [11], respectively.

Now we shall show that (3) and (4) imply (1) and (2); in fact:

$$C = \sum_{k=0}^{\infty} \frac{4+2k-1}{(2k+1)(2k+2)(2k+3)} \cdot \frac{\zeta(2k)}{4^k} = 4A + B,$$
$$D = \sum_{k=0}^{\infty} \frac{3+2k-1}{(2k+1)(2k+2)(2k+3)} \cdot \frac{\zeta(2k)}{4^k} = 3A + B,$$

Whose solution is $A = C - D = -\frac{5}{8\pi^2}\zeta(3)$ and $B = 4D - C = \frac{3}{4\pi^2}\zeta(3)$, in agreement with (1) and (2), q.e.d. Besides:

$$E \equiv \sum_{k=0}^{\infty} \frac{2k}{(2k+1)(2k+2)(2k+3)} \cdot \frac{\zeta(2k)}{4^k} = \sum_{k=0}^{\infty} \frac{1+2k-1}{(2k+1)(2k+2)(2k+3)} \cdot \frac{\zeta(2k)}{4^k} = A + B = \frac{1}{8\pi^2}\zeta(3), \quad (5)$$

Similarly:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+2)(2k+3)} \cdot \frac{\zeta(2k)}{4^k} = \sum_{k=0}^{\infty} \frac{2+2k-1}{(2k+1)(2k+2)(2k+3)} \cdot \frac{\zeta(2k)}{4^k} = 2A + B = -\frac{1}{2\pi^2}\zeta(3), \quad (6)$$

This last series representation was deduced by Zhang-Williams [11].

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