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Some Expressions For Stirling Numbers of The Second Kind

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Abstract: We obtain some relations for Stirling numbers of the second kind.

Key words: Combinatorial relations - Stirling numbers

INTRODUCTION

In [1] it was studied the identity of Charalambides [2, 3]:

$$\sum_{r=k}^{n} S_r^{(k)}\, S_{n+1}^{[r+1]} = \binom{n}{k}, \quad n \geq k \geq 1, \tag{1}$$

Involving the Stirling numbers of the first and second kind [4-7], and it was proved that it implies the property:

$$\sum_{r=k}^{n} r\, S_r^{(k)}\, S_{n+1}^{[r+1]} = \binom{n}{k-1}. \tag{2}$$

It is clear that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, then from (1) and (2):

$$\sum_{r=k}^{n} S_r^{(k)} \left(k\, S_{n+1}^{[r+1]} - n\, r\, S_n^{[r+1]} \right) = 0. \tag{3}$$

Besides, we know the relation $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, thus from (1):

$$2^n = \sum_{r=0}^{n} S_{n+1}^{[r+1]} \sum_{k=0}^{r} S_r^{(k)} = S_{n+1}^{[1]}\, S_0^{(0)} + S_{n+1}^{[2]}\, S_1^{(1)} = 1 + S_{n+1}^{[2]}\,, \tag{4}$$

Because [4, 8, 9]:

$$\sum_{k=0}^{r} S_r^{(k)} = 0, \ \ r \geq 2\,, \tag{5}$$

Hence (4) gives the following values [4, 9-11]:

$$S_{n+1}^{[2]} = 2^n - 1, \quad n \geq 1. \tag{6}$$

Similarly, $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$, then from (1):

$$3^n = \sum_{r=0}^{n} S_{n+1}^{[r+1]} \sum_{k=0}^{r} 2^k\, S_r^{(k)} = S_{n+1}^{[1]} + S_{n+1}^{[2]} \sum_{k=0}^{1} 2^k\, S_1^{(k)} + S_{n+1}^{[3]} \sum_{k=0}^{2} 2^k\, S_2^{(k)}\,, \tag{7}$$

Because [4, 8, 12, 13]:

$$\sum_{k=0}^{r} 2^k\, S_r^{(k)} = 0, \ \ r \geq 3, \tag{8}$$

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Thus (7) implies the expression [4, 9, 11, 14]:

$$S_{n+1}^{[3]} = \frac{1}{2}\,(3^n - 2^{n+1} + 1), \quad n \geq 2. \qquad (9)$$

In this manner we can obtain the values [11]:

$$3!\; S_{n+1}^{[4]} = 4^n - 3^{n+1} + 3 \cdot 2^n - 1, n \geq 3; \;\; 4!\; S_{n+1}^{[5]} = 5^n - 4^{n+1} + 2 \cdot 3^{n+1} - 2^{n+2} + 1, n \geq 4, \dots \qquad (10)$$

On the other hand, from (2):

$$\binom{n}{k-2} = \sum_{r=k-1}^{n} r\; S_r^{(k-1)}\, S_{n+1}^{[r+1]} = \sum_{r=k-1}^{n} r\left(S_{r+1}^{(k)} + r\, S_r^{(k)}\right) S_{n+1}^{[r+1]},$$

$$= \sum_{r=k}^{n} r^2\; S_r^{(k)}\, S_{n+1}^{[r+1]} - \sum_{j=k}^{n+1} S_{n+1}^{[j]}\; S_j^{(k)} + \sum_{j=k}^{n+1} j\; S_{n+1}^{[j]}\, S_j^{(k)}, \qquad (11)$$

But we have the orthogonality $\sum_{j=k}^{n+1} S_{n+1}^{[j]}\; S_j^{(k)} = \delta_{k,n+1} = 0, n \geq k,$ and the Akiyama-Tanigawa's identity [10, 15, 16]:

$$\sum_{j=k}^{n+1} j\; S_j^{(k)}\, S_{n+1}^{[j]} = (-1)^{n+k+1}\binom{n+1}{k-1}, \qquad (12)$$

Then (11) gives the relation:

$$\sum_{r=k}^{n} r^2\; S_r^{(k)}\, S_{n+1}^{[r+1]} = \binom{n}{k-2}\left[1 + (-1)^{n+k}\,\frac{n+1}{k-1}\right], \quad n \geq k \geq 2. \qquad (13)$$

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