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On the Structure of Lanczos Potential in Geometries of Petrov Types III, O and N

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Abstract: We show the spinor and Newman-Penrose versions of the Lanczos generator for the Weyl tensor in arbitrary geometries of Petrov types O, N and III.

Key words: Conformal tensor · Canonical null tetrad · Lanczos potential · Petrov types

INTRODUCTION

The Lanczos potential $K_{\mu\nu\alpha}$ [1-4] verifies the algebraic symmetries:

$$K_{\mu\nu\alpha} = -K_{\nu\mu\alpha}, \quad K_{\mu\nu\alpha} + K_{\nu\mu\alpha} + K_{\alpha\mu\nu} = 0, \quad K_{\nu}^{\mu\nu} = 0, \tag{1}$$

and it generates the conformal tensor [5] via the expression:

$$C_{\mu\nu\alpha\beta} = K_{\mu\nu\alpha;\beta} - K_{\alpha\beta\mu;\nu} - K_{\alpha\beta\mu;\nu} + \frac{1}{2}[(K_{\mu\beta} + K_{\beta\mu})g_{\nu\alpha} + (K_{\nu\alpha} + K_{\alpha\nu})g_{\mu\beta} - (K_{\mu\alpha} + K_{\alpha\mu})g_{\nu\beta} - (K_{\nu\beta} + K_{\beta\nu})g_{\mu\alpha}], \tag{2}$$

If we select [6, 7]:

$$K_{\mu\nu\alpha} = \frac{1}{3}(2F_{\mu\nu;\alpha} + F_{\alpha\nu;\mu} - F_{\alpha\mu;\nu} + F_{\nu\lambda}^{;\lambda}\, g_{\alpha\mu} - F_{\mu\lambda}^{;\lambda}\, g_{\alpha\nu}), \tag{3}$$

for arbitrary $F_{\mu\nu} = -F_{\nu\mu}$:

$$S_{\mu\nu\alpha} \equiv K_{\mu\nu\alpha} + i * K_{\mu\nu\alpha} = \frac{1}{3}(2S_{\mu\nu;\alpha} + S_{\alpha\nu;\mu} - S_{\alpha\mu;\nu} + S_{\nu\lambda}^{;\lambda}\, g_{\alpha\mu} - S_{\mu\lambda}^{;\lambda}\, g_{\alpha\nu}), \tag{4}$$

such that $S_{\mu\nu} \equiv F_{\mu\nu} + i * F_{\mu\nu}$ with the participation of the dual tensors $*K_{\mu\nu\alpha} = \frac{1}{2}\eta_{\mu\nu\lambda\beta}K^{\lambda\beta}{}_{\alpha}$ and $*F_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu\lambda\beta}F^{\lambda\beta}$.

From (4) the corresponding Lanczos spinor [8, 9] is given by:

$$3L_{ABCD} = \nabla_{AD}\ \varphi_{CA} + \nabla_{CD}\ \varphi_{AB}, \tag{5}$$

which implies the following equations in the Newman-Penrose (NP) formalism [10-12]:

$$\begin{aligned}
&\Omega_0 = D\phi_0 + 2(-\varepsilon\phi_0 + \kappa\phi_1), \quad 3\Omega_1 = \bar{\delta}\phi_0 + 2[D\phi_1 - (\alpha+\pi)\phi_0 + \rho\phi_1 + \kappa\phi_2],\\
&\Omega_3 = \bar{\delta}\phi_2 + 2(-\lambda\phi_1 + \alpha\phi_2), \quad 3\Omega_2 = D\phi_2 + 2[\bar{\delta}\phi_1 - \lambda\phi_0 - \pi\phi_1 + (\rho+\varepsilon)\phi_2],\\
&\Omega_4 = \delta\phi_0 + 2(-\beta\phi_0 + \sigma\phi_1), \quad 3\Omega_5 = \Delta\phi_0 + 2[\delta\phi_1 - (\gamma+\mu)\phi_0 + \tau\phi_1 + \sigma\phi_2],\\
&\Omega_7 = \Delta\phi_2 + 2(-\nu\phi_1 + \gamma\phi_2), \quad 3\Omega_6 = \delta\phi_2 + 2[\Delta\phi_1 - \nu\phi_0 - \mu\phi_1 + (\beta+\tau)\phi_2],
\end{aligned} \tag{6}$$

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for the NP components of Lanczos potential in terms of the spin coefficients and the NP projections of $F_{\mu\nu}$

The work [13] used the canonical null tetrad [5, 10, 14] to determine the NP components of $K_{\mu\nu\alpha}$, that is, a solution of the Wey-Lanczos equations [8]:

$$\Omega_0 = q\kappa, \quad \Omega_3 = -q\lambda, \quad \Omega_4 = q\sigma, \quad \Omega_7 = -q\nu,$$
$$\Omega_1 = \frac{q}{3}\rho, \quad \Omega_2 = -\frac{q}{3}\pi, \quad \Omega_5 = \frac{q}{3}\tau, \quad \Omega_6 = -\frac{q}{3}\mu, \tag{7}$$

For arbitrary spacetimes with Petrov types III and N for $q = 1$ and $q = \frac{1}{2}$, respectively. It is simple see that the relations (6) imply (7) if $\phi_0 = \phi_2 = 0$ and $\phi_1 = \frac{q}{2}$, therefore:

$$F_{\mu\nu} = q(\eta_\nu \; l_\nu - n_\nu \; l_\mu), \quad S_{\mu\nu} = qM_{\mu\nu} = q(m_\mu \overline{m}_\nu - m_\nu \overline{m}_\mu + n_\mu \; l_\nu - n_\nu \; l_\mu), \tag{8}$$

Hence (3) is a Lanczos potential if we employ (8) in the corresponding canonical null tetrad for the types N and III. For any conformally flat space we can apply the expressions (6) into the Weyl-Lanczos equations to obtain 0 = 0, that is, (3) with arbitrary $F_{\mu\nu}$ is a Lanczos generator for any spacetime of Petrov type O.

The possible physical meaning of $K_{\mu\nu\alpha}$ and its structure for any 4-space with Petrov types I, II and D, are open problems.

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