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Chebyshev Polynomials and Stirling Interpolation

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Abstract: We apply operators of central differences to the Chebyshev polynomials to obtain combinatorial identities.

Key words: Stirling interpolation % Chebyshev polynomials % Stirling functions

INTRODUCTION

In the Stirling interpolation [1-4] participate the operators (and * defined by:

$$\delta f(x) = f(x+1)\text{-}2\,f(x) + f(x-1), \qquad \gamma f(x) = \frac{1}{2}\,[f(x+1)\text{-}f(x-1)], \tag{1}$$

such that:

$$[\gamma \delta^k f(x)](0) = \frac{1}{2(k+1)}\left[\delta^{k+1}\big(x f(x)\big)\right](0), \qquad \left[\delta^j g(x)\right](0) = \sum_{r=0}^{2j} (-1)^r \binom{2j}{r} g(j\text{-}r), \tag{2}$$

in particular, for arbitrary functions *h*(*x*) [even] and *p* (*x*) [odd]:

$$[\gamma \delta^q h(x)](0) = 0, \qquad [\delta^q p(x)](0) = 0, \qquad q \geq 0. \tag{3}$$

The eigenfunctions of the operator * are given by [3]:

$$\delta\; cos(x\theta) = -4\, sin^2\left(\frac{\theta}{2}\right) \cdot cos(x\theta), \qquad \delta\; \frac{sin(x\theta)}{sin\theta} = -4\, sin^2\left(\frac{\theta}{2}\right) \cdot \frac{sin(x\theta)}{sin\theta}\,. \tag{4}$$

In Sec. 2 the relations (2) and (4) are applied to the Chebyshev polynomials [2, 3, 5-8] to deduce combinatorial identities.

Polynomials of Chebyshev:

From (2) and (4):

$$[\delta^k \cos(x\theta)](0) = [-4\, sin^2\left(\frac{\theta}{2}\right)]^k = \sum_{r=0}^{2k} (-1)^r \binom{2k}{r} \cos[(k\text{-}r)\theta], \tag{5}$$

or in terms of *z* = *cos 2*:

$$2\sum_{q=1}^{k} (-1)^q \binom{2k}{k\text{-}q} T_q(z) = [2(1-z)]^k \text{-} \binom{2k}{k}, \qquad k = 1, 2, 3, \ldots \tag{6}$$

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where $T_n(z) = T_n(\cos\theta) = \cos(n\theta)$ are the Chebyshev polynomials of the first kind, thus a simple iterative process gives the expressions [2, 3, 7]:

$$T_1(z) = z\,, \quad T_2(z) = 2\,z^2 - 1\,, \quad T_3(z) = 4\,z^3 - 3\,z\,, \quad T_4(z) = 8\,z^4 - 8\,z^2 + 1\,, \ldots \tag{7}$$

The known property $T_m(1) = 1, \forall\, m \geq 1$ and (6) imply the following combinatorial identity:

$$\sum_{k=1}^{n} (-1)^k \binom{2n}{n-k} = -\frac{1}{2}\binom{2n}{n}. \tag{8}$$

The function $\cos(x\theta)$ accepts the infinite Stirling expansion [3]:

$$\cos(x\theta) = \sum_{k=0}^{\infty} \left[-4\sin^2\left(\frac{\theta}{2}\right)\right]^k \Phi_{2k}(x), \tag{9}$$

with the participation of the Stirling functions [9]:

$$\Phi_{2k}(x) = \frac{x\,(x+k-1)!}{(2k)!\,(x-k)!} = \frac{1}{2}\left[\binom{x+k}{2k} + \binom{x+k-1}{2k}\right], \quad k = 0, 1, 2, \ldots \tag{10}$$

then from (9) and (10) for $x = n$:

$$T_n(z) = n\sum_{k=0}^{n} \frac{(n+k-1)!}{(2k)!\,(n-k)!}\,[2(z-1)]^k\,. \tag{11}$$

We have the relations $T_{2m}(0) = (-1)^m$ and $T_{2m+1}(0) = 0$, thus (11) gives the combinatorial identities:

$$\sum_{k=0}^{2m} (-2)^k \frac{(2m+k-1)!}{(2k)!\,(2m-k)!} = \frac{(-1)^m}{2m}, \qquad \sum_{k=0}^{2m+1} (-2)^k \frac{(2m+k)!}{(2k)!\,(2m+1-k)!} = 0. \tag{12}$$

Similary, from (2) and (4):

$$\left[\gamma\delta^k\,\frac{\sin(x\theta)}{\sin\theta}\right](0) = \left[-4\sin^2\left(\frac{\theta}{2}\right)\right]^k = \frac{1}{2(k+1)}\sum_{r=0}^{2k+2} (-1)^r \binom{2k+2}{r}(k+1-r)\frac{\sin[(k+1-r)\theta]}{\sin\theta}, \tag{13}$$

that is:

$$\sum_{r=0}^{k} (-1)^r \binom{2k+2}{k-r}(r+1)\,U_r(z) = (k+1)\,[2(1-z)]^k, \tag{14}$$

where $U_n(z) = U_n(\cos\theta) = \frac{\sin[(n+1)\theta]}{\sin\theta}$ are the Chebyshev polynomials of the second kind [7], then from (14) are immediate the expressions:

$$U_1(z) = 2\,z, \quad U_2(z) = 4\,z^2 - 1, \quad U_3(z) = 8\,z^3 - 4\,z, \quad U_4(z) = 16\,z^4 - 12\,z^2 + 1, \ldots \tag{15}$$

The property $U_n(1) = n + 1$ and (14) imply the combinatorial identity:

$$\sum_{r=0}^{k} (-1)^r \binom{2k+2}{k-r}(r+1)^2 = 0, \quad k \geq 1. \tag{16}$$

The function $\frac{\sin(x\theta)}{\sin\theta}$ admits the infinite Stirling interpolation [3]:

$$\frac{\sin(x\theta)}{\sin\theta} = \sum_{k=0}^{\infty} [-4\sin^2\left(\frac{\theta}{2}\right)]^k \ \Phi_{2k+1}(x), \tag{17}$$

with the Stirling functions [9]:

$$\Phi_{2k+1}(x) = \frac{2\,(k+1)\cdot(x+k)!}{(2k+2)!\,(x-k-1)!}, \qquad k \geq 0, \tag{18}$$

then from (17) and (18) for $x = n + 1$:

$$U_n(z) = \sum_{k=0}^{n} \binom{n+k+1}{2k+1} [2(z-1)]^k. \tag{19}$$

We know the values $U_{2m} = (-1)^m$ and $U_{2m+1}(0) = 0$, then (19) generates the combinatorial identities:

$$\sum_{k=0}^{2m} (-2)^k \binom{2m+k+1}{2k+1} = (-1)^m, \qquad \sum_{k=0}^{2m+1} (-2)^k \binom{2m+k+2}{2k+1} = 0. \tag{20}$$

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