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Chebyshev's Associated Polynomials

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Abstract: We know that if the operator $\frac{d^N}{dx^N}$ is applied to the Legendre polynomials we obtain their associated polynomials, then here we show that this process can be employed for the first-kind Chebyshev polynomials $T_n(x)$ to construct new polynomials $T_n^m(x)$ in terms of the Gauss hypergeometric function.

Key words: Gauss hypergeometric function · Chebyshev polynomials

INTRODUCTION

The first-kind Chebyshev polynomials T_n, (x), $|x| \leq 1$, verify the differential equation [1-5]:

$$(1-x^2)\frac{d^2}{dx^2}T_n - x\frac{d}{dx}T_n + n^2 T_n = 0, \quad n = 0,1,2,... \tag{1}$$

which is equivalent to the following expression in terms of the Gauss hypergeometric function [6-8]:

$$T_n(x) = {}_2F_1\left(-n,\ n; \frac{1}{2}; \frac{1-x}{2}\right). \tag{2}$$

On the other hand, we know the property:

$$\frac{d^N}{dx^N}\ {}_2F_1(a,b;c;z) \propto\ {}_2F_1(a+N,\ b+N;\ \ c+N;\ z), \tag{3}$$

then in the next Section we shall apply the operator $\frac{d^N}{dx^N}$ to (2) for the case $N = n - m$ to obtain $T_n^m(x)$, polynomials of degree m, which allow to construct the four types of Chebyshev polynomials [1-5, 9].

Associated Polynomials of Chebyshev: We apply $\frac{d^{n-m}}{dx^{n-m}}$ to (2) and we use (3) with an adequate factor of proportionality, to deduce the polynomials:

$$T_n^m(x) = (-1)^m \binom{2n-m}{m}\ {}_2F_1\left(-m,\ 2n-m;\ n-m+\frac{1}{2}; \frac{1-x}{2}\right),\ \ m = 0,1,...,n \tag{4}$$

verifying the differential equation:

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$$(1-x^2)\frac{d^2}{dx^2}T_n^m-(2n-2m+1)\,x\frac{d}{dx}\,T_n^m+m(2n-m)\,T_n^m=0. \qquad (5)$$

It is simple to calculate these associated polynomials of Chebyshev, for example:

$T_3^1=-5x,\ T_3^2=8x^2-2,\ T_5^2=32x^2-4,\ T_5^3=-56x^3+21x,\ \ T_5^4=48x^4-36x^2+3,$ etc.

The equation (1) is obtained from (5) for the case $m = n$, then (4) implies the following connection for the first-kind $T_n(x)=(-1)^n\ T_n^n(x);$; besides, it is easy to show that the associated polynomials (4) can generate the other types of Chebyshev polynomials [4, 9-11]:

$$U_n(x)=\frac{2(-1)^n}{2+n}\ T_{n+1}^n(x),\ \ V_n(x)=\frac{(-1)^n}{n+1}\ \ T_{2n+1}^{2n}\left(\sqrt{\frac{1-x}{2}}\right),\ \ W_n(x)=\frac{1}{n+1}\ T_{2n+1}^{2n}\left(\sqrt{\frac{1+x}{2}}\right). \qquad (6)$$

The expression is equivalent to [8]:

$$T_n^m(x)=2^{m-1}\frac{(n-1)!(2n-m)}{m!(n-m)!}\sum\nolimits_{k=0}^{m}(-1)^{k-m}\binom{m}{k}\ {}_2F_1(k-m,-1-2m;-2m;1)x^k. \qquad (7)$$

The polynomials T_n^m can be interpreted as coefficients in the characteristic equation of the Chebyshev matrices [12].

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