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Identities of Munarini and Simons via Gauss Hypergeometric Function

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Abstract: We employ the Gauss hypergeometric function to show the binomial identities of Simons and Munarini.

Key words: Hypergeometric functions · Binomial identities

INTRODUCTION

From [1] we have the following property of the Gauss hypergeometric function [2]:

$$ {}_2F_1(-n, b-a-n; 1-a-n; 1-z) = \frac{(-1)^n (1-b)_n}{(a)_n} (1-z)^{a+n} {}_2F_1(a, b; b-n; z), \tag{1} $$

where we can use the values $a = -n$, $b = a + 1$ and $z = x + 1$ to deduce the relation:

$$ {}_2F_1(-n, a+1; 1; -x) = (-1)^n \binom{a}{n} {}_2F_1(-n, a+1; a-n+1; x+1), \tag{2} $$

and for the case $a = n$

$$ {}_2F_1(-n, n+1; 1; -x) = (-1)^n {}_2F_1(-n, n+1; 1; x+1). \tag{3} $$

In Sec. 2 we show that (2) and (3) imply the binomial identities of Munarini [3, 4] and Simons [5], respectively.

Simons and Munarini Identities: First, we study the quantity:

$$ A \equiv \sum_{k=0}^{\infty} q_k, \qquad q_k = \binom{n}{k}\binom{a+k}{k} x^k, \tag{4} $$

therefore:

$$ \frac{q_{k+1}}{q_k} = \frac{(k-n)(k+a+1)}{k+1} \cdot \frac{(-x)}{k+1}, $$

which means that A can be written in terms of the Gauss hypergeometric function [6]:

$$ A = {}_2F_1(-n, a+1; 1; -x). \tag{5} $$

On the other hand:

$$ B \equiv (-1)^n \binom{a}{n} \sum_{k=0}^{\infty} p_k, \quad p_k = \frac{\binom{a}{n-k}\binom{a+k}{k}}{\binom{a}{n}} (-1)^k (x+1)^k, \tag{6} $$

thus:

$$ \frac{p_{k+1}}{p_k} = \frac{(k-n)(k+a+1)}{k+a-n+1} \cdot \frac{x+1}{k+1}, $$

hence [6]:

$$ B = (-1)^n \binom{a}{n} {}_2F_1(-n, a+1); a-n+1; x+1), \tag{7} $$

then (2), (5) and (7) imply the Munarini's identity [3, 4]:

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$$\sum_{k=0}^{n}\binom{n}{k}\binom{a+k}{k}x^k=\sum_{k=0}^{n}\binom{a}{n-k}\binom{a+k}{k}(-1)^{n-k}(x+1)^k, \tag{8}$$

and if $a = n$ we obtain the Simons formula [5]:

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}(x+1)^k, \tag{9}$$

which is equivalent to (3).

Munarini generalized (8), in fact:

$$\sum_{k=0}^{n}\binom{\beta-a+n}{n-k}\binom{\beta+k}{k}(-y)^{n-k}(x+y+1)^k=\sum_{k=0}^{n}\binom{a}{n-k}\binom{\beta+k}{k}y^{n-k}(x+1)^k, \tag{10}$$

and it is possible to analyze (10) via ${}_2F_1$; for $\beta = \alpha$ and $y = -1$ the identity (10) implies (8).

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