On the Resolvent of a Matrix

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Abstract: We employ the Faddeev-Sominsky method to deduce the Lanczos expression for the resolvent of a matrix.

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INTRODUCTION

For an arbitrary matrix $A_{nm}$ ($A'$) its characteristic polynomial [1-3]:

$$
\rho(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n, \quad (1)
$$
can be obtained, through several procedures [1, 4-8].

The approach of Leverrier-Takeno [4, 9-13] is a simple and interesting technique to construct (1) based in the traces of the powers $A^r$, $r = -1, \ldots, n$.

On the other hand, it is well known that an arbitrary matrix satisfies its characteristic equation, which is the Cayley-Hamilton-Frobenius identity [1-3, 14]:

$$
A^n + a_1 A^{n-1} + \ldots + a_{n-1} A + a_n I = 0. \quad (2)
$$

If $A$ is non-singular (that is, det $A \neq 0$), then from (2) we obtain its inverse matrix:

$$
A^{-1} = -\frac{1}{a_n} (A^{-1} + a_1 A^{n-2} + \ldots + a_{n-1} I) \quad (3)
$$

where $a_n \neq 0$ because $a_n = (-1)^n \det A$.

Faddeev-Sominsky [15-24] proposed an algorithm to determine $A^{-1}$ in terms of $A'$ and their traces, which is equivalent [23] to the Cayley-Hamilton-Frobenius theorem (2) plus the Leverrier-Takeno’s method to construct the characteristic polynomial of a matrix $A$. In Sec. 2, we employ the Faddeev-Sominsky’s procedure to obtain the Lanczos expression [25] for the resolvent of $A$ [20, 21, 26, 27], that is, the Laplace transform of $\exp(t A)$ [28].

Leverrier-Takeno and Faddeev-Sominsky Techniques: If we define the quantities:

$$
a_0 = 1, \quad s_k = \text{tr} A^k, k = 1, 2, \ldots, n \quad (4)
$$

then the process of Leverrier-Takeno [4, 9-13] implies (1) wherein the $a_i$ are determined with the Newton’s recurrence relation:

$$
r a_r + s_1 a_{r-1} + s_2 a_{r-2} + \ldots + s_{r-1} a_1 + s_r = 0, \quad r = 1, 2, \ldots, n \quad (5)
$$

therefore:

$$
a_1 = -s_1, 2! \ a_2 = (s_1)^2 - s_2, \ 3! \ a_3 = -(s_1)^3 + 3s_1s_2 - 2s_3

4! \ a_4 = (s_1)^4 - 6(s_1)^2 s_2 + 8s_1s_3 + 3(s_2)^2 - 6s_4, \ etc \quad (6)
$$
in particular, det $A = (1 - \cdots a_n$), that is, the determinant of any matrix only depends on the traces $s$, which means that $A$ and its transpose have the same determinant. In [29, 30] we find the general expression:

$$
\begin{bmatrix}
  s_1 & k - 1 & 0 & \cdots & 0 \\
  s_2 & s_1 & k - 2 & \cdots & 0 \\
  s_{k-1} & s_{k-2} & \cdots & \cdots & 1 \\
  s_{k} & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  s_{n} & \cdots & \cdots & \cdots & s_{n}
\end{bmatrix}, \quad k = 1, \ldots, n
$$

(7)

The Faddeev-Sominsky’s procedure [15-24] to obtain $A^{-1}$ is a sequence of algebraic computations on the powers $A'$ and their traces, in fact, this algorithm is given via the instructions:

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By synthetic division of two polynomials [1]:

$$p(z) = \sum_{r=0}^{n-1} (\lambda^r + a_1 \lambda^{r-1} + a_2 \lambda^{r-2} + \ldots + a_{n-r} \lambda + a_r) z^{n-r},$$

then under the change $\lambda = -A$ we obtain the Lanczos expression for the resolvent of a matrix [20, 21, 26, 27]:

$$\frac{1}{zI - A} = \frac{1}{p(z)} \sum_{r=0}^{n-1} z^{n-1-r} C_r = \frac{Q(z)}{p(z)}; \quad (14)$$

if $A$ is non-singular, then (14) for $z = 0$ implies (9). McCarthy [31] used (14) and the Cauchy’s integral theorem in complex variable to show the Cayley-Hamilton-Frobenius identity indicated in (2); the relation (14) is the Laplace transform of $\exp(tA)$ [28].

On the other hand, Sylvester [32-35] obtained the following interpolating definition of $f(A)$:

$$f(A) = \sum_{j=1}^{n} f(\lambda_j) \prod_{k \neq j} \frac{A - \lambda_k I}{\lambda_j - \lambda_k}, \quad (15)$$

which is valid if all eigenvalues are different from each other. Buchheim [36] generalized (15) to multiple proper values using Hermite interpolation, thereby giving the first completely general definition of a matrix function. From (14) and (15) for $f(s) = \frac{1}{z-s}$ we deduce the properties:

$$Q(z) = \sum_{k=1}^{n} \prod_{k=1, k \neq j}^{n} \frac{z-\lambda_k}{\lambda_j - \lambda_k}(A-\lambda_k I),$$

$$Q j \vec{u}_j = \prod_{k=1, k \neq j}^{n} (\lambda_j - \lambda_k)\vec{u}_j, \quad (16)$$

hence the eigenvectors of $A$ showed in (11) also are proper vectors of the matrices $Q_i$. Besides, from (11) and (16):

$$A Q j \vec{u}_j = \prod_{k=1, k \neq j}^{n} (\lambda_j - \lambda_k)\lambda_j \vec{u}_j = \lambda_j Q j \vec{u}_j \quad : \quad A Q j = \lambda_j Q j \quad \text{that is, each column of } Q \text{ is eigenvector of } A \text{ with proper value } \lambda_j.$$

The resolvent (14) implies the relation $(A - z I) Q (z) = -p(z) I$, then $(A - \lambda_j I) Q (\lambda_j) = -p(\lambda_j) I = 0$ in accordance with (17).
From the Sylvester’s formula (15) with \( f(z) = p(z) \) we obtain \( p(A) = 0 \), which is the Cayley-Hamilton-Frobenius theorem indicated in (2). If \( f(z) = e^z \), then (15) allows to construct \( \exp(tA) \) that, in particular, is valuable to determine the motion of classical charged particles into a homogeneous electromagnetic field, and to integrate the Frenet-Serret equations with constant curvatures [37]. In [34, 38] we find that the book of Frazer-Duncan-Collar [39] emphasizes the important role of the matrix exponential in solving differential equations and was the first to employ matrices as an engineering tool, and indeed the first book to treat matrices as a branch of applied mathematics.

REFERENCES