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On the Resolvent of a Matrix

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Abstract: We employ the Faddeev-Sominsky method to deduce the Lanczos expression for the resolvent of a matrix.

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INTRODUCTION

For an arbitrary matrix A_{nxn} (A^i_j) its characteristic polynomial [1-3]:

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n, \tag{1}$$

can be obtained, through several procedures [1, 4-8]. The approach of Leverrier-Takeno [4, 9-13] is a simple and interesting technique to construct (1) based in the traces of the powers A^r, r =-1,...,n

On the other hand, it is well known that an arbitrary matrix satisfies its characteristic equation, which is the Cayley-Hamilton-Frobenius identity [1-3, 14]:

$$A^n + a_1 A^{n-1} + ... a_{n-1}A + a_n I = 0. \tag{2}$$

If A is non-singular (that is, det A ≠ 0), then from (2) we obtain its inverse matrix:

$$A^{-1} = -\frac{1}{a_n}(A^{-1} + a_1 A^{n-2} + ... + a_{n-1} I) \tag{3}$$

where $a_n \neq 0$ because $a_n = (-1)^n$ det A.

Faddeev-Sominsky [15-24] proposed an algorithm to determine A^{-1} in terms of A^r and their traces, which is equivalent [23] to the Cayley-Hamilton-Frobenius theorem (2) plus the Leverrier-Takeno's method to construct the characteristic polynomial of a matrix A. In Sec. 2, we employ the Faddeev-Sominsky's procedure to obtain the Lanczos expression [25] for the resolvent of A [20, 21, 26, 27], that is, the Laplace transform of exp(t A) [28].

Leverrier-Takeno and Faddeev-Sominsky Techniques: If we define the quantities:

$$a_0 = 1, \quad s_k = trA^k, k = 1,2,...,n \tag{4}$$

then the process of Leverrier-Takeno [4, 9-13] implies (1) wherein the a_i are determined with the Newton's recurrence relation:

$$ra_r + s_1 a_{r-1} + s_2 a_{r-2} + ... + s_{r-1}a_1 + s_r = 0, \quad r = 1,2,...,n \tag{5}$$

therefore:

$$a_1 = -s_1, 2!\ a_2 = (s_1)^2 - s_2, \quad 3!\ a_3 = -(s_1)^3 + 3s_1 s_2 - 2s_3$$
$$4!\ a_4 = (s_1)^4 - 6(s_1)^2 s_2 + 8s_1 s_3 + 3(s_2)^2 - 6s_4,\ etc \tag{6}$$

in particular, det A = (1 –)^n a_n, that is, the determinant of any matrix only depends on the traces s_r, which means that A and its transpose have the same determinant. In [29, 30] we find the general expression:

$$a_k = \frac{(-1)^k}{k!}\begin{vmatrix} s_1 & k-1 & 0 & \cdots & 0 \\ s_2 & s_1 & k-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_{k-2} & \cdots & \cdots & 1 \\ s_k & s_{k-1} & \cdots & \cdots & s_1 \end{vmatrix}, \qquad k = 1, \ldots, n. \tag{7}$$

The Faddeev-Sominsky's procedure [15-24] to obtain A^{-1} is a sequence of algebraic computations on the powers A^r and their traces, in fact, this algorithm is given via the instructions:

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$$B_1 = A, \qquad q_1 = trB_1, \qquad C_1 = B_1 - q_1 I,$$
$$B_2 = C_1 A, \qquad q_2 = \frac{1}{2} trB_2, \qquad C_2 = B_2 - q_2 I,$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$B_{n-1} = C_{n-2} A, \quad q_{n-1} = \frac{1}{n-1} trB_{n-1}, \quad C_{n-1} = B_{n-1} - q_{n-1} I,$$
$$B_n = C_{n-1} A, \qquad q_n = \frac{1}{n} trB_n, \tag{8}$$

then:

$$A^{-1} = \frac{1}{q_n} C_{n-1}. \tag{9}$$

For example, if we apply (8) for $n = 4$, then it is easy to see that the corresponding q_r imply (6) with $q_j = -a_j$ and besides (9) reproduces (3). By mathematical induction one can prove that (8) and (9) are equivalent to (3), (4) and (5), showing [23] thus that the Faddeev-Sominsky's technique has its origin in the Leverrier-Takeno method plus the Cayley-Hamilton-Frobenius theorem.

From (8) we can see that [26]:

$$C_k = A_k + a_1 A^{k-1} + a_2 A^{k-2} + \ldots + a_{k-1} A + a_k I,$$
$$k = 1,2,\ldots,n-1, \quad C_n = B_n - q_n I = 0, \tag{10}$$

and for $k = n - 1$:

$$C_{n-1} = A^{n-1} + a_1 A^{n-2} a_2 A^{n-3} + \ldots + a_{n-2} A + a_{n-1} I \overset{(3)}{=} -a_n A^{-1},$$

in harmony with (9) because $a_n = -q_n$. The property $C_n = 0$ is equivalent to (2); if A is singular, the process (8) gives the adjoint matrix of A [2, 3, 16], in fact, $Adj\ A = (-1)^{n+1} C_{n-1}$

If the roots of (1) have distinct values, then the Faddeev-Sominsky's algorithm allows obtain the corresponding eigenvectors of A [6]:

$$A\vec{u}_k = \lambda_k \vec{u}_k, \quad k = 1,2,\ldots,n, \tag{11}$$

because for a given value of *k,* each column of:

$$Q_k = \lambda_k^{n-1} I + \lambda_k^{n-2} C1 + \ldots + C_{n-1}, \tag{12}$$

satisfies (11) [16, 18, 27], and therefore all columns of Q_k are proportional to each other, that is, *rank* $Q_k = 1$ [18]; we note that $Q_k = Q(\lambda_k)$ with the participation of the matrix:

$$Q(z) \equiv z^{n-1} I + z^{n-2} C_1 + z^{n-3} C_2 + \ldots + z C_{n-2} + C_{n-1}, \tag{13}$$

By synthetic division of two polynomials [1]:

$$\frac{p(z)}{z-\lambda} = \sum_{r=0}^{n-1} (\lambda^r + a_1 \lambda^{r-1} + a_2 \lambda^{r-2} + \ldots + a_{r-1}\lambda + a_r) z^{n-1-r},$$

then under the change $\lambda \rightarrow A$ we obtain the Lanczos expression for the resolvent of a matrix [20, 21, 26, 27]:

$$\frac{1}{zI - A} = \frac{1}{p(z)} \sum_{r=0}^{n-1} z^{n-1-r} C_r = \frac{Q(z)}{p(z)}; \tag{14}$$

if A is non-singular, then (14) for $z = 0$ implies (9). McCarthy [31] used (14) and the Cauchy's integral theorem in complex variable to show the Cayley-Hamilton-Frobenius identity indicated in (2); the relation (14) is the Laplace transform of exp(t A) [28].

On the other hand, Sylvester [32-35] obtained the following interpolating definition of *f*(A):

$$f(A) = \sum_{j=1}^{n} f(\lambda_j) \prod_{k \neq j} \frac{A - \lambda_k I}{\lambda_j - \lambda_k}, \tag{15}$$

which is valid if all eigenvalues are different from each other. Buchheim [36] generalized (15) to multiple proper values using Hermite interpolation, thereby giving the first completely general definition of a matrix function. From (14) and (15) for $f(s) = \frac{1}{z-s}$ we deduce the properties:

$$Q(z) = \sum_{j=1}^{n} \prod_{k=1, k \equiv j}^{n} \frac{z - \lambda_k}{\lambda_j - \lambda_k} (A - \lambda_k I)$$
$$\therefore Q_j = \prod_{k=1, k \neq j}^{n} (A - \lambda_k I),$$

$$Q_j \vec{u}_j = \prod_{k=1, k \neq j}^{n} (\lambda_j - \lambda_k) \vec{u}_j, \tag{16}$$

hence the eigenvectors of A showed in (11) also are proper vectors of the matrices Q_j Besides, from (11) and (16):

$$A Q_j \vec{u}_j = \prod_{k=1, k \neq j}^{n} (\lambda_j - \lambda_k) \lambda_j \vec{u}_j = \lambda_j Q_j \vec{u}_j \therefore A Q_j = \lambda_j Q_j \tag{17}$$

that is, each column of Q_j is eigenvector of A with proper value λ_j The resolvent (14) implies the relation (A – z I) Q (z) = – $p(z)$ I, then (A – λ_k I) Q (λ_k) = – $p(\lambda_k)$ I = 0 in according with (17).

From the Sylvester's formula (15) with $f(z) = p(z)$ we obtain $p(A) = 0$, which is the Cayley-Hamilton-Frobenius theorem indicated in (2). If $f(z) = e^{tz}$, then (15) allows to construct exp(tA) that, in particular, is valuable to determine el motion of classical charged particles into a homogeneous electromagnetic field, and to integrate the Frenet-Serret equations with constant curvatures [37]. In [34, 38] we find that the book of Frazer-Duncan-Collar [39] emphasizes the important role of the matrix exponential in solving differential equations and was the first to employ matrices as an engineering tool, and indeed the first book to treat matrices as a branch of applied mathematics.

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