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On the Carlitz-Gessel and Jha Identities for Bernoulli Numbers

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Abstract: We study the Jha and Carlitz-Gessel identities involving Bernoulli numbers.

Key words: Sun's formula - Stirling numbers - Bernoulli polynomials - Carlitz-Gessel identity - Jha formula

INTRODUCTION

Sun [1-3] obtained the following identity involving Bernoulli polynomials [4-7]:

$$(-1)^l \sum_{r=0}^{l} \binom{l}{r} x^{l-r} B_{k+r}(z) = (-1)^k \sum_{j=0}^{k} \binom{k}{j} x^{k-j} B_{l+j}(y), \quad x+y+z=1; \quad k,l \geq 0, \tag{1}$$

Which for $x = 1,\ y = -z$ gives the expression of Kejian-Zhiwei-Hao[8, 9]:

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} B_{n+i}(z) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} B_{m+j}(-z), \tag{2}$$

where we can use $z = 0$ to obtain the Carlitz-Gessel identity [1, 2, 10-19]:

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} B_{n+i} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} B_{m+j}, \tag{3}$$

Involving Bernoulli numbers [5-7, 12, 15], whose binomial inversion [7, 20] implies the relation:

$$B_{m+n} = (-1)^{m+n} \sum_{j=0}^{n} \sum_{i=0}^{m} \binom{n}{j} \binom{m}{i} B_{i+j}. \tag{4}$$

On the other hand, we have the Jha identity [21, 22]:

$$B_{i+j} = \sum_{k=0}^{j} \sum_{r=0}^{i} \frac{(-1)^{k+r} (k!\, r!)^2}{(k+r+1)!} S_i^{[r]} S_j^{[k]}, \tag{5}$$

With the participation of Stirling numbers of the second kind, whose application into (4) gives the expression:

$$B_{m+n} = (-1)^{m+n} \sum_{k=0}^{n} \sum_{r=0}^{m} \frac{(-1)^{k+r} (k!\, r!)^2}{(k+r+1)!} S_{n+1}^{[k+1]} S_{m+1}^{[r+1]}, \tag{6}$$

where it was employed the Roman identity [7, 23, 24]:

$$\sum_{j=k}^{n} \binom{n}{j} S_j^{[k]} = S_{n+1}^{[k+1]}. \tag{7}$$

Pan-Sun [25] obtained the interesting relation:

$$\frac{(-1)^m}{m} \sum_{k=0}^{m} \binom{m}{k} B_{m-k}(x)\, B_{n-1+k}(y) - \frac{B_m(z)}{m} B_{n-1}(y) =$$

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$$x + y + z = 1, \quad m, n \geq 1 \tag{8}$$

$$= \frac{(-1)^n}{n} \sum_{r=0}^{n} \binom{n}{r} B_{n-r}(x)\, B_{m-1+r}(z) - \frac{B_n(y)}{n} B_{m-1}(z),$$

Which for $x = 1,\ y = z = 0$ implies the following Woodcock's identity[26]:

$$\frac{(-1)^m}{m} \sum_{q=0}^{m-1} \binom{m}{q} (-1)^q\, B_q\, B_{m+n-q-1} = \frac{(-1)^n}{n} \sum_{j=0}^{n-1} \binom{n}{j} (-1)^j\, B_j\, B_{m+n-j-1}\,, \tag{9}$$

where we can use $n = 1$ to deduce the known Euler's result:

$$\frac{1}{m} \sum_{r=1}^{m} \binom{m}{r} B_r B_{m-r} + B_{m-1} = -\, B_m, \qquad m \geq 1. \tag{10}$$

For $m, n \geq 2$ the expression (9) adopts the form:

$$\frac{(-1)^m}{m} \sum_{q=0}^{m-1} \binom{m}{q} B_q\, B_{m+n-q-1} = \frac{(-1)^n}{n} \sum_{j=0}^{n-1} \binom{n}{j} B_j\, B_{m+n-j-1}\,, \tag{11}$$

That for example, for $n = 2$, gives the property:

$$\frac{(-1)^m}{m} \sum_{q=0}^{m-1} \binom{m}{q} B_q\, B_{m-q+1} = \frac{1}{2} (B_{m+1} - B_m). \tag{12}$$

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