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Stirling Numbers and Riemann Zeta Function

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Abstract: We study certain identities of Srivastava-Choi involving Stirling numbers and the Riemann zeta function.

Key words: Stirling numbers - Riemann zeta function - Harmonic numbers

INTRODUCTION

Srivastava-Choi [1] obtained the following relations:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n)(a)_k} S_{k+n}^{(n)} = -\frac{(1-a)_n}{n!}\psi^{(n)}(a-n), \qquad n \geq 1,$$
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n-1)(a)_k} S_{k+n}^{(n)} = \frac{(-a)_n}{a\,(n-1)!}\psi^{(n-1)}(a-n), \qquad n \geq 2, \tag{1}$$

Involving Stirling numbers of the first kind and polygamma functions [2-7].

If in (1) we apply:

$$a = n+1, \qquad (-n)_n = (-1)^n\, n!\,, \qquad \psi^{(n)}(1) = -(-1)^n\, n!\; \zeta(n+1), \qquad n \geq 1, \tag{2}$$

We deduce two interesting expressions for the Riemann zeta function:

$$\sum_{j=n}^{\infty} \frac{(-1)^{n+j}}{j \cdot j!} S_j^{(n)} = \zeta(n+1), \qquad n \geq 1\,, \tag{3}$$

$$\sum_{j=n}^{\infty} \frac{(-1)^{n+j}}{(j-1) \cdot j!} S_j^{(n)} = \zeta(n), \qquad n \geq 2\,. \tag{4}$$

The property (3) is the Shen's identity [8-11], but we make to note that (3) and (4) also were obtained by Jordan [12] and Hansen [13]. If in (1) we employ:

$$a = n+2, \qquad \psi^{(n)}(2) = -(-1)^n\, n!\,[\,\zeta(n+1) - 1], \qquad n \geq 1, \tag{5}$$

We find the relations:

$$\sum_{j=n}^{\infty} \frac{(-1)^{n+j}}{j \cdot (j+1)!} S_j^{(n)} = \zeta(n+1) - 1, \qquad n \geq 1\,, \tag{6}$$

$$\sum_{j=n}^{\infty} \frac{(-1)^{n+j}}{(j-1) \cdot (j+1)!} S_j^{(n)} = \frac{1}{2}[\,\zeta(n) - 1], \qquad n \geq 2\,; \tag{7}$$

With (4) and the recurrence property of the Stirling numbers of the first kind [5]:

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$$S_{j+1}^{(n)} = S_j^{(n-1)} - j\, S_j^{(n)}\,, \tag{8}$$

It is immediate the identity [1, 12, 13]:

$$\sum_{k=n+2}^{\infty} \frac{(-1)^{k+n}}{(k-1)(k-2)\cdot(k-1)!} S_k^{(n+2)} = \zeta(n+1), \qquad n \geq 1. \tag{9}$$

We note that the expression [7]:

$$\psi^{(n)}(m) = (-1)^{n+1}\, n! \left[\zeta(n+1) - \sum_{r=1}^{m-1} \frac{1}{r^{n+1}} \right], \tag{10}$$

Implies the values (2) and (5). If in (3) we use $n = 2$ and [5]:

$$S_k^{(2)} = (-1)^k (k-1)!\ H_{k-1}\,, \tag{11}$$

Involving harmonic numbers, then we obtain the relation [1, 8]:

$$\sum_{k=2}^{\infty} \frac{H_{k-1}}{k^2} = \zeta(3). \tag{12}$$

We know the inversion formula [5, 6, 14, 15]:

$$\sum_{r=k}^{n} f(r)\, S_n^{(r)} = g(k) \qquad \therefore \sum_{r=k}^{n} g(r)\, S_n^{[r]} = f(k), \tag{13}$$

Whose application to (3) gives the identity:

$$\sum_{k=n}^{\infty} (-1)^k\ \zeta(k+1) S_k^{[n]} = \frac{(-1)^n}{n \cdot n!}\,, \qquad n \geq 1, \tag{14}$$

Involving Stirling numbers of the second kind [5]. If we remember that $S_k^{[1]} = 1,\ k \geq 1$, then from (14) for $n = 1$:

$$\sum_{j=2}^{\infty} (-1)^j\ \zeta(j) = 1, \tag{15}$$

Which is a particular case of the expression [1]:

$$\sum_{j=2}^{\infty} (-1)^j\ \zeta(j)\ t^{j-1} = \psi(1+t) + \gamma_0, \tag{16}$$

For $t = 1$ and $\psi(2) = 1 - \gamma_0$, where γ_0 is the Euler-Mascheroni constant [1, 16, 17].

Similarly, the relation (4) for $n = 2$ implies [5]:

$$\sum_{j=2}^{\infty} \frac{H_{j-1}}{j\,(j-1)} = \zeta(2), \tag{17}$$

and the inversion of (4) via (13) gives the identity:

$$\sum_{j=n}^{\infty} (-1)^j\ \zeta(j) S_j^{[n]} = \frac{(-1)^n}{(n-1)\cdot n!}\ , \qquad n \geq 2, \tag{18}$$

where we can employ $n = 2$ and $S_j^{[2]} = 2^{j-1} - 1,\ j \geq 2,$ to obtain:

$$\sum_{j=2}^{\infty} (-2)^j\ \zeta(j) = 3, \tag{19}$$

Which is a particular case of (16) for $t = 2$ and $\psi(3) = \frac{3}{2} - \gamma_0$. We may realize a similar study for (6) and (7).

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