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On the Colour Partitions $p_r(n)$

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Abstract: We study several relationships between partition and divisors functions.

Key words: Partitions - Sum of divisors function - Partition congruences

INTRODUCTION

We have the following result [1]:

$$B_{n,k}(1!\,a_1\,,2!\,a_2\,,\dots,(n-k+1)!\,a_{n-k+1}\,)=\frac{n!}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}\,p_r(n), \tag{1}$$

Involving the partial Bell polynomials and the quantities:

$$a_j=\begin{cases}0 & if \quad j\neq\frac{N}{2}\,(3N+1),\\ (-1)^N & if \quad j=\frac{N}{2}\,(3N+1),\end{cases}\qquad N=0,\pm1,\pm2,\dots, \tag{2}$$

and in (1) we can apply binomial inversion [2, 3] to obtain the relation:

$$p_r(n)=\frac{1}{n!}\sum_{k=0}^{r}\binom{r}{k}\,k!\,B_{n,k}\big(1!\,a_1\,,2!\,a_2\,,\dots,(n-k+1)!\,a_{n-k+1)}\,\big), \tag{3}$$

Which for $r=0,1,2,\dots$ gives the expressions:

$$p_0(0)=1,\qquad p_0(m)=0,\quad m\geq1,\qquad p_1(n)=a_n=\sum_{k=0}^{n}a_{n-k}\;p_0(k),\quad n\geq0,$$
$$p_2(n)=\sum_{k=0}^{n}a_{n-k}\,a_k=\sum_{k=0}^{n}a_{n-k}\;p_1(k),\qquad p_3(n)=\sum_{k=0}^{n}a_{n-k}\;p_2(k),\dots, \tag{4}$$

Hence it is natural the property:

$$p_{r+1}(n)=\sum_{k=0}^{n}a_{n-k}\;p_r(k); \tag{5}$$

It is simple to show (5), in fact:

$$\sum_{n=0}^{\infty}p_{r+1}(n)\,q^n=[(q;q)_\infty]^{r+1}=(q;q)_\infty[(q;q)_\infty]^r=\sum_{l=0}^{\infty}a_l\,q^l\sum_{j=0}^{\infty}p_r(j)\,q^j,$$

$$=\sum_{n=0}^{\infty}\big[\sum_{t=0}^{n}a_{n-t}\,p_r(t)\big]\,q^n\quad\Longrightarrow\quad(5).$$

From (3) for $n=0,1,2,\dots$:

$$p_r(0)=1,\ r\geq0,\qquad p_r(1)=-r,\qquad p_r(2)=\frac{r}{2!}\,(r-3),\qquad p_r(3)=-\frac{r}{3!}\,(r-1)(r-8),$$
$$p_r(4)=\frac{r}{4!}\,(r-1)(r-3)(r-14),\qquad p_r(5)=-\frac{r}{5!}\,(r-3)(r-6)(r^2-21\,r+8),\dots, \tag{6}$$

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Then (5) and (6) imply the values:

$p_r(n)$

$n \setminus r$	2	3	4	5	6	7	8	9
1	-2	-3	-4	-5	-6	-7	-8	-9
2	-1	0	2	5	9	14	20	27
3	2	5	8	10	10	7	0	-12
4	1	0	-5	-15	-30	-49	-70	-90
5	2	0	-4	-6	0	21	64	135
6	-2	-7	-10	-5	11	35	56	54
7	0	0	8	25	42	41	0	-49
8	-2	0	9	15	0	-49	-125	-189
9	-2	0	0	-20	-70	-133	-160	-85

(7)

Jha [1, 4] deduced the relation:

$$p(n) = \sum_{r=0}^{n} (-1)^r \binom{n+1}{r+1} p_r(n), \tag{8}$$

where we can employ $n = 5m + 4$ and the Ramanujan's congruence $p(5m + 4) \equiv 0 \ mod\ 5$ [5-8] to obtain the result:

$$p_{5m+4}(5m + 4) \equiv 0 \quad mod\ 5, \quad m \geq 0, \tag{9}$$

For example, (7) gives the values $p_4(4) = -5$ and $p_9(9) = -85$ in agreement with (9). We know the following property [6-8]:

$$p_{m\,w\,+\,4}\left(n\,w - \frac{w+1}{6}\right) \equiv 0 \quad mod\ w, \quad m \geq 0, \quad n \geq 1, \tag{10}$$

With w a prime number of the form $6\lambda - 1$; then (10) implies (9) for $w = 5$ and $n = m + 1$.

Osler-Hassen-Chandrupatla [9, 10] showed interesting expressions involving (2) and the sum of divisors function:

$$\sigma(m) = \sum_{j=1}^{m} j\, a_{n-j}\, p(j), \quad m \geq 1, \qquad \sigma(n) = -n\, a_n - \sum_{k=1}^{n-1} a_{n-k}\, \sigma(k), \quad n \geq 2, \tag{11}$$

Which give the relation:

$$p(n) = -a_n - \frac{1}{n} \sum_{k=1}^{n-1} k\ p_2(n-k)\, p(k), \quad n \geq 2, \quad p(0) = p(1) = 1. \tag{12}$$

It is simple to prove that:

$$a_{5m+4} = a_{11m+6} = 0, \quad m \geq 0, \tag{13}$$

Then from (12):

$$\sum_{k=1}^{5m+3} k\, p_2(5m + 4 - k)\, p(k) \equiv 0 \ mod\ 5\,, \quad \sum_{k=1}^{11m+5} k\, p_2(11m + 6 - k)\, p(k) \equiv 0 \quad mod\ 11\,. \tag{14}$$

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