Computational and Applied Mathematical Sciences 6 (2): 33-34, 2021
ISSN 2222-1328

DOI: 10.5829/idosi.cams.2021.33.34

On the Hermite Polynomials

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Abstract: We give elementary deductions of identities obtained by Qi-Guo involving the Hermite polynomials.

Key words: Hermite polynomials - Qi-Guo's identities

INTRODUCTION

The Hermite polynomials [1-7] have the following property [8, 9]:

$$H_n(y-x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (2x)^{n-k} H_k(y), \tag{1}$$

Which for $y = x$ implies the recurrence relation obtained by Qi-Guo [10]:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (2x)^{n-k} H_k(x) = H_n(0) = \begin{cases} 0, & n \text{ is odd}, \\ \frac{(-1)^m (2m)!}{m!}, & n = 2m. \end{cases} \tag{2}$$

We know the expression [11-13]:

$$H_m(x)\, H_n(x) = m!\, n! \sum_{r=0}^{\min\,(m,n)} \frac{2^r}{(m-r)!\,(n-r)!\,r!} H_{m+n-2r}(x), \tag{3}$$

Then for $m = n$:

$$H_n^2(x) = \sum_{r=0}^{n} 2^r\, r! \binom{n}{r}^2 H_{2(n-r)}(x), \tag{4}$$

where we can employ the formula [8]:

$$H_N(x) = N! \sum_{t=0}^{[\frac{N}{2}]} \frac{(-1)^t}{N!\,(N-2t)!} (2x)^{N-2t}, \tag{5}$$

With $N = 2(n-r)$ and thus to deduce a identity for the square of any Hermite polynomial:

$$H_n^2(x) = n!\; 2^n \sum_{q=0}^{n} \frac{(-4)^q\, q!}{(2q)!} \left[\sum_{j=q}^{n} \left(-\frac{1}{2}\right)^j \binom{2j}{j} \binom{j}{q} \binom{n}{j} \right] x^{2q}, \tag{6}$$

Which is equivalent to the expression obtained by Qi-Guo [10]:

$$H_n^2(x) = (-2)^n\, n! \sum_{k=0}^{n} \frac{(-2)^k}{k!} \left[\sum_{r=0}^{n-k} \frac{1+(-1)^r}{2} \cdot \frac{(r-1)!!}{r!!} \binom{n-r-1}{k-1} \right] x^{2k}. \tag{7}$$

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