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Relationships Between the Sum of Divisors and Partition Functions Via Determinants

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Abstract: We show how the partition and divisor functions can be connected by determinants.

Key words: Partitions – Determinants - Sum of divisors

INTRODUCTION

Gould [1] exhibits the following result:

$$If \quad e_n = \sum_{j=0}^{n} b_j^{(n)}\, c_{n-j}\,, \qquad n \geq 0, \qquad b_0^{(k)} \neq 0, \tag{1}$$

Then

$$c_n = \frac{1}{\prod_{r=0}^{n} b_0^{(r)}} \begin{vmatrix} e_n & b_1^{(n)} & b_2^{(n)} & b_3^{(n)} & \cdots & b_n^{(n)} \\ e_{n-1} & b_0^{(n-1)} & b_1^{(n-1)} & b_2^{(n-1)} & \cdots & b_{n-1}^{(n-1)} \\ e_{n-2} & 0 & b_0^{(n-2)} & b_1^{(n-2)} & \cdots & b_{n-2}^{(n-2)} \\ e_{n-3} & 0 & 0 & b_0^{(n-3)} & \cdots & b_{n-3}^{(n-3)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ e_0 & 0 & 0 & 0 & \cdots & b_0^{(0)} \end{vmatrix}, \tag{2}$$

with interesting applications.

Jha [2, 3] obtained the relation:

$$p(n) = \sum_{r=0}^{n} (-1)^r \binom{n+1}{r+1} p_r(n), \qquad n \geq 0, \tag{3}$$

With the structure (1):

$$\frac{p(n)}{(n+1)!} = \sum_{j=0}^{n} \frac{p_{n-j}(n)}{j!}\, \frac{(-1)^{n-j}}{(n-j+1)!}, \quad e_n = \frac{p(n)}{(n+1)!}, \quad b_j^{(n)} = \frac{p_{n-j}(n)}{j!}, \quad c_{n-j} = \frac{(-1)^{n-j}}{(n-j+1)!}, \tag{4}$$

Then (2) and (4) imply the connection:

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$$\frac{(-1)^n}{(n+1)!}=\frac{1}{\prod_{r=0}^{n} p_r(r)}\begin{vmatrix} \frac{p(n)}{(n+1)!} & \frac{p_{n-1}(n)}{1!} & \frac{p_{n-2}(n)}{2!} & \frac{p_{n-3}(n)}{3!} & \cdots & \cdots & \frac{p_0(n)}{n!} \\ \frac{p(n-1)}{n!} & \frac{p_{n-1}(n-1)}{0!} & \frac{p_{n-2}(n-1)}{1!} & \frac{p_{n-3}(n-1)}{2!} & \cdots & \cdots & \frac{p_0(n-1)}{(n-1)!} \\ \frac{p(n-2)}{(n-1)!} & & \frac{p_{n-2}(n-2)}{0!} & \frac{p_{n-3}(n-2)}{1!} & \cdots & \cdots & \frac{p_0(n-2)}{(n-2)!} \\ \vdots & 0 & 0 & \frac{p_{n-3}(n-3)}{0!} & \cdots & \cdots & \frac{p_0(n-3)}{(n-3)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \frac{p_0(n-4)}{(n-4)!} \\ \frac{p(3)}{4!} & 0 & 0 & 0 & \cdots & \cdots & \vdots \\ \frac{p(2)}{3!} & 0 & 0 & 0 & \cdots & \frac{p_1(2)}{1!} & \frac{p_0(2)}{2!} \\ \frac{p(1)}{2!} & 0 & 0 & 0 & \cdots & \frac{p_1(1)}{0!} & \frac{p_0(1)}{1!} \\ \frac{p(0)}{1!} & & 0 & 0 & \cdots & \cdots & \frac{p_0(0)}{0!} \end{vmatrix}. \tag{5}$$

For example, from (5) for $n = 4$:

$$\frac{1}{5!}=-\frac{1}{25}\begin{vmatrix} \frac{p(4)}{5!} & \frac{p_3(4)}{1!} & \frac{p_2(4)}{2!} & \frac{p_1(4)}{3!} & \frac{p_0(4)}{4!} \\ \frac{p(3)}{4!} & \frac{p_3(3)}{0!} & \frac{p_2(3)}{1!} & \frac{p_1(3)}{2!} & \frac{p_0(3)}{3!} \\ \frac{p(2)}{3!} & 0 & \frac{p_2(2)}{0!} & \frac{p_1(2)}{1!} & \frac{p_0(2)}{2!} \\ \frac{p(1)}{2!} & 0 & 0 & \frac{p_1(1)}{0!} & \frac{p_0(1)}{1!} \\ p(0) & 0 & 0 & 0 & p_0(0) \end{vmatrix}=-\frac{1}{25}\begin{vmatrix} \frac{5}{5!} & 0 & \frac{1}{2!} & 0 & 0 \\ \frac{3}{4!} & 5 & \frac{2}{1!} & 0 & 0 \\ \frac{2}{3!} & 0 & \frac{-1}{0!} & \frac{-1}{1!} & 0 \\ \frac{1}{2!} & 0 & 0 & \frac{-1}{0!} & 0 \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix}=\frac{1}{120}.$$

Jha [2] deduced the expression:

$$\sigma(n)=n\sum_{r=0}^{n}\frac{(-1)^r}{r}\binom{n}{r}p_r(n), \qquad n\ge 1, \qquad \sigma(0)=0, \tag{6}$$

With $lim_{\ r\to 0}\ \frac{p_r(n)}{r}=0$, and (6) can be written in the form (1):

$$\frac{\sigma(n)}{n!\,n}=\sum_{j=0}^{n}\frac{p_{n-j}(n)}{j!}\,\frac{(-1)^{n-j}}{(n-j)!\,(n-j)}, \qquad e_n=\frac{\sigma(n)}{n!\,n}, \qquad b_j^{(n)}=\frac{p_{n-j}(n)}{j!}, \qquad c_{n-j}=\frac{(-1)^{n-j}}{(n-j)!\,(n-j)}, \tag{7}$$

Therefore:

$$\frac{(-1)^n}{n!\,n}=\frac{1}{\prod_{r=0}^{n} p_r(r)}\begin{vmatrix} \frac{\sigma(n)}{n!\,n} & \frac{p_{n-1}(n)}{1!} & \frac{p_{n-2}(n)}{2!} & \frac{p_{n-3}(n)}{3!} & \cdots & \cdots & \frac{p_0(n)}{n!} \\ \frac{\sigma(n-1)}{(n-1)!\,(n-1)} & \frac{p_{n-1}(n-1)}{0!} & \frac{p_{n-2}(n-1)}{1!} & \frac{p_{n-3}(n-1)}{2!} & \cdots & \cdots & \frac{p_0(n-1)}{(n-1)!} \\ \frac{\sigma(n-2)}{(n-2)!\,(n-2)} & & \frac{p_{n-2}(n-2)}{0!} & \frac{p_{n-3}(n-2)}{1!} & \cdots & \cdots & \frac{p_0(n-2)}{(n-2)!} \\ \vdots & 0 & 0 & \frac{p_{n-3}(n-3)}{0!} & \cdots & \cdots & \frac{p_0(n-3)}{(n-3)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \frac{p_0(n-4)}{(n-4)!} \\ \frac{\sigma(3)}{3!\,3} & 0 & 0 & 0 & \cdots & \cdots & \vdots \\ \frac{\sigma(2)}{2!\,2} & 0 & 0 & 0 & \cdots & \frac{p_1(2)}{1!} & \frac{p_0(2)}{2!} \\ \frac{\sigma(1)}{1!} & 0 & 0 & 0 & \cdots & \frac{p_1(1)}{0!} & \frac{p_0(1)}{1!} \\ 0 & & 0 & 0 & \cdots & \cdots & \frac{p_0(0)}{0!} \end{vmatrix}. \tag{8}$$

From (8) for $n = 4$:

$$\frac{1}{4!\,4} = -\frac{1}{25}\begin{vmatrix} \frac{7}{96} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{4}{18} & 5 & 2 & 0 & 0 \\ \frac{3}{4} & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{96}.$$

Osler-Hassen-Chandrupatla [4] showed the property:

$$\sigma(n) = \sum_{k=0}^{n} k\ a_{n-k}\, p(k) = \sum_{j=0}^{n} a_j\,(n-j)\,p(n-j), \tag{9}$$

where:

$$a_j = \begin{cases} 0 & if \quad j \neq \frac{N}{2}\,(3N+1), \\ (-1)^N & if \quad j = \frac{N}{2}\,(3N+1), \end{cases} \qquad N = 0, \pm 1, \pm 2, \dots, \tag{10}$$

Then (1), (2) and (9) imply the determinant:

$$n\,p(n) = \begin{vmatrix} \sigma(n) & a_1 & a_2 & a_3 & \cdots & \cdots & \cdots & a_n \\ \sigma(n-1) & 1 & a_1 & a_2 & \cdots & \cdots & \cdots & a_{n-1} \\ \sigma(n-2) & 0 & 1 & a_1 & a_2 & \cdots & \cdots & a_{n-2} \\ \vdots & \vdots & 0 & 1 & a_1 & \cdots & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & 1 & \ddots & & \vdots \\ \sigma(1) & 0 & 0 & 0 & \vdots & \ddots & & a_1 \\ \sigma(0) & 0 & 0 & 0 & 0 & \cdots & \cdots & 1 \end{vmatrix}. \tag{11}$$

From (11) for $n = 4$:

$$4\,p(4) = \begin{vmatrix} 7 & -1 & -1 & 0 & 0 \\ 4 & 1 & -1 & -1 & 0 \\ 3 & 0 & 1 & -1 & -1 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 20 \qquad \therefore \qquad p(4) = 5,$$

and for $n = 5$:

$$5\,p(5) = \begin{vmatrix} 6 & -1 & -1 & 0 & 0 \\ 7 & 1 & -1 & -1 & 0 \\ 4 & 0 & 1 & -1 & -1 \\ 3 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix} = 35 \qquad \therefore \qquad p(5) = 7.$$

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