Computational and Applied Mathematical Sciences 6 (2): 27-29, 2021
ISSN 2222-1328

DOI: 10.5829/idosi.cams.2021.27.29

On the Jha's Identity for the Sum of Inverses of Odd Divisors of a Positive Integer

[1]*P. Lam-Estrada and* [2]*J. López-Bonilla*

[1]ESFM, Instituto Politécnico Nacional, Edif. 9, 3er. Piso,
CP 07738, CDMX, México,
[2]ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: Jha showed an interesting relation involving the number of representations of a positive integer *n* as a sum of *k* squares and the sum of reciprocals of odd divisors of *n*. Here we study this Jha's relation via partial and complete Bell polynomials.

Key words: Bell polynomials - Sum of squares - Jacobi's theta function - Sum of divisors

INTRODUCTION

Jha [1] obtained the following results for $n = 1, 2, \dots$:

$$x_n \equiv -2 \cdot n! \sum_{odd\ d|n} \frac{1}{d} = \sum_{k=1}^{n} (-1)^{k-1} (k-1)!\ B_{n,k}\left(\theta^{(1)}(0), \theta^{(2)}(0), \dots, \theta^{(n-k+1)}(0)\right), \qquad (1)$$

$$B_{n,k}\left(\theta^{(1)}(0), \dots, \theta^{(n-k+1)}(0)\right) = \frac{(-1)^n\, n!}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} r_j(n), \qquad (2)$$

Involving the odd divisors of *n*, the partial Bell polynomials [2-13], the number of representations of *n* as a sum of squares such that representations with different orders and distinct signs are counted as different [14], and the theta function:

$$[\theta(q)]^k = \sum_{n=0}^{\infty} (-1)^n\, r_k(n)\, q^n, \qquad \theta(q) = \prod_{j=1}^{\infty} \frac{1-q^j}{1+q^j} = \sum_{k=-\infty}^{\infty} (-1)^k\, q^{k^2}, \quad |q| < 1, \qquad (3)$$

Thus (1) and (2) imply the interesting connection:

$$\sum_{odd\ d|n} \frac{1}{d} = \frac{1}{2} \sum_{j=1}^{n} \frac{(-1)^{n-j}}{j} \binom{n}{j} r_j(n). \qquad (4)$$

It is immediate the inversion of (1) [15]:

$$\theta^{(n)}(0) = B_n(x_1, x_2, \dots, x_n), \qquad (5)$$

That is [2, 10, 13]:

$$\theta^{(n)}(0) = \begin{vmatrix} \binom{n-1}{0} x_1 & \binom{n-1}{1} x_2 & \cdots & \binom{n-1}{n-2} x_{n-1} & \binom{n-1}{n-1} x_n \\ -1 & \binom{n-2}{0} x_1 & \cdots & \binom{n-2}{n-3} x_{n-2} & \binom{n-2}{n-2} x_{n-1} \\ 0 & -1 & \cdots & \binom{n-3}{n-4} x_{n-3} & \binom{n-3}{n-3} x_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{1}{0} x_1 & \binom{1}{1} x_2 \\ 0 & 0 & \cdots & -1 & \binom{0}{0} x_1 \end{vmatrix}, \qquad (6)$$

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México. E-mail: jlopezb@ipn.mx.

With the recurrence relation:

$$\theta^{(n)}(0) = \sum_{k=1}^{n} \binom{n-1}{k-1} x_k \, \theta^{(n-k)}(0), \quad n \geq 1, \qquad \theta^{(0)}(0) = 1. \tag{7}$$

From (2) we can obtain the expression:

$$y_n \equiv \sum_{j=1}^{n} (-1)^{n-j} \binom{n+1}{j+1} r_j(n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, k! \; B_{n,k}(\theta^{(1)}(0), \dots, \theta^{(n-k+1)}(0)), \tag{8}$$

Which means [4, 16-19]:

$$\sum_{n=0}^{\infty} y_n \, t^n = \frac{1}{\sum_{j=0}^{\infty} \theta^{(j)}(0) \frac{t^j}{j!}}, \quad y_0 \equiv 1, \tag{9}$$

With the Hessenberg determinant:

$$y_n = (-1)^n \begin{vmatrix} \frac{\theta^{(1)}(0)}{1!} & 1 & 0 & \cdots & 0 \\ \frac{\theta^{(2)}(0)}{2!} & \frac{\theta^{(1)}(0)}{1!} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ \frac{\theta^{(n)}(0)}{n!} & \frac{\theta^{(n-1)}(0)}{(n-1)!} & \cdots & \cdots & \frac{\theta^{(1)}(0)}{1!} \end{vmatrix}, \quad n \geq 1, \tag{10}$$

O well:

$$\theta^{(n)}(0) = (-1)^n \, n! \begin{vmatrix} y_1 & 1 & 0 & \cdots & 0 \\ y_2 & y_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ y_n & y_{n-1} & \cdots & \cdots & y_1 \end{vmatrix}, \tag{11}$$

Verifying the recurrence relation:

$$\theta^{(n)}(0) = -\sum_{k=0}^{n-1} \frac{n!}{k!} \, \theta^{(k)}(0) \, y_{n-k}, \quad n \geq 1. \tag{12}$$

To check the expressions (6), (7), (10), (11) and (12) we have the values:

$$x_1, x_2, x_3, x_4, x_5, x_6, \dots = -2, -4, -16, -48, -288, -5760, \dots, \quad \text{respectively,}$$
$$y_1 = 2, \; y_2 = 4, \; y_3 = 8, \dots, \quad \theta^{(1)}(0) = -2, \; \theta^{(2)}(0) = \theta^{(3)}(0) = 0, \; \theta^{(4)}(0) = 48, \dots \tag{13}$$

REFERENCES

1. Kumar Jha, S., 2021. An identity for the sum of inverses of odd divisors of n in terms of the number of representations of n as a sum of squares, Rocky Mountain J. Maths., 51(2): 581-583.
2. Johnson, W.P., 2002. The curious history of Faà di Bruno's formula, The Math. Assoc. of America 109: 217-234.
3. Riordan, J., 1968. Combinatorial identities, John Wiley and Sons, New York.
4. Comtet, L., 19714. Advanced combinatorics, D. Reidel Pub., Dordrecht, Holland.
5. Zave, D.A., 1976. A series expansion involving the harmonic numbers, Inform. Process. Lett., 5(3): 75-77.
6. Chen, X. and W. Chu, 2009. The Gauss ${}_2F_1(1)$-summation theorem and harmonic number identities, Integral Transforms and Special Functions, 20(12): 925-935.
7. Connon, D.F., 2010. Various applications of the (exponential) complete Bell polynomials, http://arxiv.org/ftp/arxiv/papers/1001/1001.2835.pdf 16 Jan 2010.
8. Quaintance, J. and H.W. Gould, 2016. Combinatorial identities for Stirling numbers, World Scientific, Singapore.
9. López-Bonilla, J., S. Vidal-Beltrán and A. Zúñiga-Segundo, 2018. Characteristic equation of a matrix via Bell polynomials, Asia Mathematika, 2(2): 49-51.

10. López-Bonilla, J., R. López-Vázquez and S. Vidal-Beltrán, 2018. Bell polynomials, Prespacetime, 9(5): 451-453.
11. López-Bonilla, J., S. Vidal-Beltrán and A. Zúñiga-Segundo, 2018. On certain results of Chen and Chu about Bell polynomials, Prespacetime Journal, 9(7): 584-587.
12. López-Bonilla, J., S. Vidal-Beltrán and A. Zúñiga-Segundo, 2018. Some applications of complete Bell polynomials, World Eng. & Appl. Sci. J., 9(3): 89-92.
13. López-Bonilla, J., R. López-Vázquez and S. Vidal-Beltrán, 2019. Stirling numbers and Bell polynomials, Prespacetime J., 10(2): 214-218.
14. Grosswald, E., 1985. Representation of integers as sums of squares, Springer-Verlag, New York.
15. https://en.wikipedia.org/wiki/Bell-polynomials
16. Vein, R. and P. Dale, 1999. Determinants and their applications in Mathematical Physics, Springer-Verlag, New York.
17. Gould, H.W., 2010. Combinatorial identities. Table I: Intermediate techniques for summing finite series, Edited and compiled by J. Quaintance.
18. Birmajer, D., J.B. Gil and M.D. Weiner, 2014. Linear recurrence sequences and their convolutions via Bell polynomials, arXiv: 1405.7727v2 [math.CO] 26 Nov 2014.
19. Shattuck, M., 2017. Some combinatorial formulas for the partial r-Bell polynomials, Notes on Number Theory and Discrete Maths., 23(1): 63-76.