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Expressions of Glaisher and Jha to Obtain the Sum of Inverses of Odd Divisors of an Integer

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Abstract: We exhibit the formulae obtained by Jha and Glaisher to determine the sum of odd divisors of a positive integer, trying to obtain a connection between the corresponding representations.

Key words: Odd divisors of an integer - Sum of squares - Stirling numbers

INTRODUCTION

Jha [1] obtained the following relation for the sum of inverses of odd divisors of a positive integer *n*:

$$2(-1)^n \sum_{odd\ d|n} \frac{1}{d} = \sum_{k=1}^{n} \frac{(-1)^k}{k} \binom{n}{k} r_k(n), \tag{1}$$

where $r_k(n)$ is the number of representations of n as a sum of k squares such that representations with different orders and distinct signs are counted as different [2-5]. On the other hand, Glaisher [6, 7] deduced an expression similar to (1):

$$2(-1)^n \sum_{odd\ d|n} \frac{1}{d} = \sum_{k=1}^{n} \frac{(-1)^k}{k} R_k(n), \tag{2}$$

In terms of other type of representations:

$$R_k(n) = 2^k P_k(n), \tag{3}$$

Being $P_k(n)$ the number of compositions of n as the sum of exactly k nonvanishing squares. For example, for $n = 8$:

$$R_k(8) = 0, \quad k = 1, 3, 4, 6, 7, \qquad R_2(8) = 4, \qquad R_5(8) = 160, \qquad R_8(8) = 256, \tag{4}$$

$$r_1(8) = 0,\ r_4(8) = 2\, r_3(8) = 6\, r_2(8) = 24, r_5(8) = 200, r_6(8) = 1020,\ r_7(8) = 3444,\ r_8(8) = 9328.$$

It is interesting to investigate some connection between the representations used by Jha and Glaisher, such that (2) implies (1); here we propose the following relationship:

$$R_k(n) = \binom{n}{k} \left[r_k(n) - \sum_{j=0}^{n-1} A_j(n)\, k^{n-j} \right], \qquad n \geq 2, \tag{5}$$

where the $A_j(n)$ are rational coefficients; thus we can apply (5) into (2) to obtain:

$$2(-1)^n \sum_{odd\ d|n} \frac{1}{d} = \sum_{k=1}^{n} \frac{(-1)^k}{k} \binom{n}{k} r_k(n) - (-1)^n\, n! \sum_{j=0}^{n-1} A_j(n)\, S_{n-1-j}^{[n]}, \tag{6}$$

Involving the Stirling numbers of the second kind [8], however, $(n-1-j) < n$ for $0 \leq j \leq n-1$, then $S_{n-1-j}^{[n]} = 0$, thus (6) gives (1), q.e.d.

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We study (5) for $n = 2, 3, \dots, 8$ with $1 \le k \le n$ to deduce expressions:

$$R_k(2) = \binom{2}{k} r_k(2), \qquad R_k(3) = \binom{3}{k} r_k(3), \qquad R_k(4) = \binom{4}{k}\left[r_k(4) - \tfrac{k^2}{12}(k^2 - 9\,k + 26)\right],$$

$$R_k(5) = \binom{5}{k}\left[r_k(5) - \tfrac{k^2}{15}(k^3 - 13\,k^2 + 59\,k - 47)\right],$$

$$R_k(6) = \binom{6}{k}\left[r_k(6) - \tfrac{k^2}{30}(k^4 - 18\,k^3 + 121\,k^2 - 252\,k + 148)\right], \tag{7}$$

$$R_k(7) = \binom{7}{k}\left[r_k(7) - \tfrac{4\,k^2}{315}(k^5 - 24\,k^4 + 226\,k^3 - 846\,k^2 + 1285\,k - 642)\right],$$

$$R_k(8) = \binom{8}{k}\left[r_k(8) - \tfrac{k^2}{3360}(13k^6 - 402k^5 + 5054\,k^4 - 28740\,k^3 + 77357\,k^2 - 91818\,k + 38536)\right];$$

We do not have a general formula for the coefficients $A_j(n)$ in (5).

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