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On the Jha and Melanfant Formulae for the Partition Function

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Abstract: We give short deductions of the Jha and Melanfant expressions for the partition function.

Key words: Partition function - Melanfant and Jha formulas

INTRODUCTION

We know the following result [1, 2] for the partition function $p(n)$:

$$\sum_{n=0}^{\infty} p(n)\, t^n = \frac{1}{\sum_{r=0}^{\infty} a_r\, t^r}\ , \qquad p(0) = a_0 = 1, \tag{1}$$

where:

$$a_j = \begin{cases} 0 & if \quad j \neq \frac{N}{2}\,(3N + a), \\ (-1)^N & if \quad j = \frac{N}{2}\,(3N+1), \end{cases} \qquad N = 0, \pm 1, \pm 2, \dots , \tag{2}$$

Therefore [3-5]:

$$p(n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k\, k!\, B_{n,k}(1!\, a_1, 2!\, a_2, \dots, (n-k+1)!\, a_{n-k+1}), \tag{3}$$

Expression recently obtained by Jha [6], involving the partial Bell polynomials [3, 7], with the recurrence relation [4]:

$$\sum_{k=0}^{n} a_k\, p(n-k) = 0, \tag{4}$$

Discovered by MacMahon [2].

On the other hand, from [8, 9] we have that relations type (1) are equivalent to the following Hessenberg determinant:

$$p(n) = (-1)^n \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & \cdots & a_1 \end{vmatrix}, \tag{5}$$

Obtained by Malenfant [10].

We note that the definition (2) gives the values:

$$a_j = \begin{cases} 1, & j = 0, 5, 7, 22, 26, 51, 57, 92, 100, 145, 155, \dots \\ -1, & j = 1, 2, 12, 15, 35, 40, 70, 77, 117, 126, 176, \dots \\ 0 & otherwise \end{cases} \tag{6}$$

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